Motives of Commutative Groups Schemes and Relative 1-Motives

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Zusammenfassung

In dieser Dissertation präsentiren wir einige neue Beitragen zur Theorie von gemischten Motiven. Genauer gesagt studieren wir die Fälle von relativen 1-Motiven und von Motiven von kommutativen Gruppeschemata, im Kontext von den triangulierten Kategorien von gemischten Motiven von Voevodsky.

Sei S ein Scheme. Voevodsky vorgestellte eine triangulierte Kategorie $\mathbf{DA}(S)$ von gemischten Motiven mit $\mathbb{Q}$-Koeffizienten über S. Diese Kategorie ist gebaut und studiert mit der Hoffnung, dass sie eine geeignete Annäherung an die derivierte Kategorie einer mutmaßlichen abelschen Kategorie $\mathbf{MM}(S)$ von gemischten Motiven über S ist. Die Erwartung ist, dass $\mathbf{DA}(S)$ eine *motivische t-Struktur* trägt, deren Herz diese mutmaßliche abelsche Kategorie $\mathbf{MM}(S)$ würde. Das ist ganz unbekannt; trotzdem kann man für das System von Kategorien $\mathbf{DA}(-)$ unmutmaßlichen ein "Formalismus der sechs Operationen " [Ayo07a] [Ayo07b] und "Realisierung Funktoren" von $\mathbf{DA}(S)$ nach Kategorien von klassichen Koeffiziensystem (konstruktible Garben, ℓ-adische Garben) [Ayo10b] [Ayo14a] konstruieren, die mit den Formalismen der sechs Operationen in der Betti und ℓ-adic Fälle kompatibel sind, so dass $\mathbf{DA}(S)$ bereits sich wie eine "derivierte Kategorie der motivischen Garben über S" verhält.

Wichtige Beispiele von Motiven in $\mathbf{DA}(S)$ sind von kommutativen Gruppenschemata über S konstruiert, und fast alle unsere Ergebnisse haben mit sie zu tun. Sei G/S ein solches glattes kommutatives Gruppenschema. Dann können wir zwei natürliche Motive von G gebaut: erst, $\Sigma^\infty G_{\mathbb{Q}} \in \mathbf{DA}(S)$, dass heißt, "$G$ als Garbe von $\mathbb{Q}$-Vektorraüme", und $M_S(G)$, dass heisst, der homologische Motiv von G als S-schema. In Kapitel 2, geschrieben mit Giuseppe Ancona und Annette Huber, werden die Zwei kompariert, und wir zeigen die folgende kanonische "Künneth Zersetzung" des Motives $M_S(G)$.

$$M_S(G) \xrightarrow{\sim} \left(\bigoplus_{n\geq 0}^{\mathrm{kd}(G/S)} \mathsf{Sym}^n \Sigma^\infty G_{\mathbb{Q}} \right) \otimes M_S(\pi_0(G/S)) \ .$$

Von diesen Folgen erwarten wir, dass der Motiv $\Sigma^\infty G_{\mathbb{Q}}$ einen Paradebeispiel für einen relativ homologische 1-Motiv ist, das heißt, ein Motiv in der Triangulierten Unterkategorie durch die relative Homologie von Kurven über S erzeugt. Das motiviert die systematische Studie von diesen Unterkategorie.

In Kapitel 3.1, definieren wir für jeden $n \in \mathbb{N}$ die Kategorie $\mathbf{DA}_n(S)$ von *homologische n-Motive* (resp. $\mathbf{DA}^n(S)$ von *kohomologische n-Motive*) als die volle Unterkategorie von $\mathbf{DA}(S)$ die ist bei homologische Motive von glatten (resp. kohomologische Motive von projektiven) S-Schemata mit relativer Dimension weniger als n erzeugt. Wir studieren seine allgemeine Eigenschaften und sein Verhalten unter den sechs Operationen. Dann zeigen wir allerdings in Kapitel 3.2, dass $\Sigma^\infty G_{\mathbb{Q}}$ in $\mathbf{DA}_1(S)$ liegt.

In Kapitel 3.3, definieren wir für jeden kohomologische Motiv M seine "1-motivische Annäherung" $\omega^1 M \in \mathbf{DA}^1(S)$. Der Funktor ω^1 heisst der *motivisch Picard Funktor*. Wir berechnen ω^1 in einem wichtige Sonderfall.

In Kapitel 3.4, definieren wir eine t-Struktur $t_{\mathbf{MM},1}$ über $\mathbf{DA}_1(S)$. Wir zeigen einigen Eigenschaften von $t_{\mathbf{MM},1}$, die zusammen nahelegen, dass $t_{\mathbf{MM},1}$ die Restriktion auf $\mathbf{DA}_1(S)$ von der mutmaßlichen motivischen t-Struktur über $\mathbf{DA}(S)$ ist. Wir studieren weiter das Herz $\mathbf{MM}_1(S)$ und wir konnektieren $\mathbf{MM}_1(S)$ mit einer früheren Theorie von 1-Motive, nähmlich, Delignes Theorie von 1-Motiven [Del74].

Abstract

In this dissertation, we present several new results in the theory of mixed motives. More precisely, we study the case of relative 1-motives and motives of commutative group schemes in the context of the triangulated categories of mixed motives of Voevodsky.

Given a scheme S, Voevodsky introduced a triangulated category $\mathbf{DA}(S)$ of mixed motives over S. This category is built and studied with the hope that it is a suitable approximation to the derived category of a conjectural abelian category $\mathbf{MM}(S)$ of mixed motives over S. The expectation is that $\mathbf{DA}(S)$ carries a *motivic t-structure* whose heart would be this conjectural abelian category. One can construct unconditionally a "six operations formalism" [Ayo07a] [Ayo07b] for the system of categories $\mathbf{DA}(-)$ and realisation functors to derived categories of systems of coefficients (constructible sheaves, l-adic sheaves) [Ayo10b] [Ayo14a] which are compatible with the classical theory of the six operations in the Betti and ℓ-adic contexts, so that $\mathbf{DA}(S)$ already behaves to a large extend like a "derived category of motivic sheaves" .

Important examples of motives in $\mathbf{DA}(S)$ are constructed out of commutative group schemes over S, and almost all our results are concerned with these. Let G/S be such a smooth commutative group scheme. Then we have two natural motives built from G: first, $\Sigma^\infty G_\mathbb{Q} \in \mathbf{DA}(S)$, which is "$G$ seen as a sheaf of $\mathbb{Q}$-vector spaces", and $M_S(G)$, that is, the homological motive of G as an S-scheme. In Chapter 2, written in collaboration with Giuseppe Ancona and Annette Huber, we compare the two, and we prove the following canonical "Künneth decomposition" of the motive $M_s(G)$.

$$M_S(G) \xrightarrow{\sim} \left(\bigoplus_{n \geq 0}^{\mathrm{kd}(G/S)} \mathsf{Sym}^n \Sigma^\infty G_\mathbb{Q} \right) \otimes M_S(\pi_0(G/S)) \; .$$

From this result, we can expect that the motive $\Sigma^\infty G_\mathbb{Q}$ should be a prime example of a relative homological 1-motive, that is, a motive in the triangulated subcategory generated by the relative homology of curves over S. This motivates the systematic study of this subcategory.

In Chapter 3.1, we define more generally for every $n \in \mathbb{N}$ the category $\mathbf{DA}_n(S)$ of *homological n-motives* (resp. $\mathbf{DA}^n(S)$ of *cohomological n-motives*) as the full triangulated subcategory of $\mathbf{DA}(S)$ which is generated by homological motives of smooth (resp. cohomological motives of projective) S-schemes of relative dimension less than n. We study their general properties and their behaviour under the six operations. Then we show in Chapter 3.2 that $\Sigma^\infty G_\mathbb{Q}$ lies indeed in $\mathbf{DA}_1(S)$.

In Chapter 3.3, we define for any cohomological motive M its "1-motivic approximation" $\omega^1 M \in \mathbf{DA}^1(S)$. The functor ω^1 is called the *motivic Picard functor*. We then compute ω^1 in an important special case.

In Chapter 3.4, we define a t-structure $t_{\mathbf{MM},1}$ on $\mathbf{DA}_1(S)$. We study some properties of $t_{\mathbf{MM},1}$, which together suggest, that $t_{\mathbf{MM},1}$ is the restriction to $\mathbf{DA}_1(S)$ of the conjectural motivic t-structure on $\mathbf{DA}(S)$. We then study further the heart $\mathbf{MM}_1(S)$ and we connect it to an existing classical theory of 1-motives, namely, Deligne's theory of 1-motives from [Del74].

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A substantial part of this work (Chapter 2) has been written in collaboration with Giuseppe Ancona and Annette Huber, builing on their previous work with Stephen Enright-Ward. I was very happy that they asked me to join their project; working with them allowed me to clarify key ideas for my work on 1-motives. I thank them both heartily.

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Contents

Chapter 1

Introduction

1.1 General introduction

This thesis presents some contributions to the theory of mixed motivic sheaves, in particular to the study of 1-motives and of motives of commutative group schemes.

The notion of the motive of an algebraic variety or more generally of a morphism of schemes has been suggested by Grothendieck as the "common motive" of many known cohomological invariants in arithmetic geometry (Betti, De Rham, Hodge, l-adic, crystalline, etc.). Although a tantalizing picture has been developed and used as a guideline in arithmetic geometry ever since, the main actor of the story, namely the abelian category of (mixed) motives, remains undefined. We refer the reader to [And04], [Kah12] and [CDb, Introduction] for more in-depth overviews than this introduction.

The relative 1-motives studied in this thesis are thus "geometric incarnations" of the relative cohomology of families of (open, singular) curves over general base schemes. The leitmotiv of this work is that relative 1-motives are closely related to commutative group schemes.

1.1.1 Pure motives of curves and abelian varieties

We start by briefly recalling the results on pure motives (i.e., motives attached to smooth projective varieties) which are at the root of our work. When C is a smooth projective curve over a field k, it is well known that the geometry of the Jacobian $J(C)$ (an abelian variety over k) can be used to recover all the standard cohomological invariants of C. For instance, the tangent space to $J(C)(\mathbb{C})$ recovers the Betti homology $H_1(C(\mathbb{C}),\mathbb{Z})$ (for $k=\mathbb{C}$)), the Zariski tangent space and the space of algebraic 1-forms of $J(C)$ control the Hodge filtration of $H^1_{dR}(C/k)$ (for k of characteristic 0) and the l-adic Tate module of $J(C)$ controls the l-adic homology $H_1(C,\mathbb{Z}_l)$ (for ℓ prime different from $\mathrm{char}(k)$). In any reasonable theory the motive of C should be closely related to $J(C)$. Moreover, the Albanese map $C\to J(C)$ should relate the motive of C to the motive of $J(C)$ itself. Furthermore, for a general smooth projective variety X/k, the dual theories of the Picard and Albanese variety realize parts of the cohomology of X in terms of the rational points/divisors on abelian varieties, and they should also fit in a motivic framework.

This is precisely what happens in Grothendieck's category $\mathbf{Chow}(k)$ of pure (Chow) motives over k with rational coefficients. Grothendieck himself proved that the subcategory of $\mathbf{Chow}(k)$ generated by motives of smooth projective curves is equivalent to the category of abelian varieties up to isogeny, with the equivalence sending $M(C)$ to $J(C)$. Moreover, Shermenev [Šer74] proved that the motive of $J(C)$ in $\mathbf{Chow}(k)$ has a decomposition mirroring the decomposition by cohomological degrees (the so-called Chow-Künneth decomposition), and that the H_1 part was directly related to the motive of C via the Albanese map. This result was later generalized to a Chow-Künneth decomposition of the motive of any abelian variety by Deninger and Murre [DM91]. In a related direction, for a general smooth projective variety, Murre [Mur90] showed how to relate parts of the Chow motive of X to the Albanese and Picard abelian varieties of X.

Our main results extend these works (and subsequent work of Orgogozo, Barbieri-Viale-Kahn, and Ancona-Huber-Enright-Ward discussed below) to the setting of triangulated categories of mixed motives over general base schemes.

1.1.2 Mixed motives à la Voevodsky

Given a scheme S, Voevodsky introduced a triangulated category $\mathbf{DM}(S)$ of mixed motives over S. This category is built and studied with the hope that it is a suitable approximation to the derived category of a conjectural abelian category $\mathbf{MM}(S)$ of mixed motives over S. The expectation is that $\mathbf{DM}(S)$ carries a motivic t-structure whose heart would be this conjectural abelian category. One can construct unconditionally a "six operations formalism" for the system of categories $\mathbf{DM}(-)$ [Ayo07a] [Ayo07b] and realisation functors to derived categories of systems of coefficients (constructible sheaves, l-adic sheaves) [Ayo10b] [Ayo14a] which are compatible with the classical theory of the six operations in the Betti and ℓ-adic contexts, so that $\mathbf{DM}(S)$ already behaves to a large extend like a "derived category of motivic sheaves" . In fact the motivic t-structure, if it exists, is determined uniquely by the requirement that it is compatible with the standard t-structures via the realisations functors. Another important point is that one expects to have at least two motivic t-structures, the standard one described above and a perverse one compatible with the classical theories of perverse sheaves.

More precisely, one should mention that there are many closely related variants of $\mathbf{DM}(S)$ in the literature and that they do not all share the set of properties sketched above. Several of them will play a role in what follows. These variants can be classified according to the following alternatives: effective/stable, with/without transfers, Nisnevich/étale topology, various boundedness and constructibility conditions, various rings of coefficients ($\mathbb{Z}$, $\mathbb{Z}[1/p]$, $\mathbb{Q}$). More details on the terminology will be given in the background section below. In this thesis, we focus on $\mathbf{DA}^{\text{ét}}(S)$, the stable unbounded category of étale motives without transfers and with rational coefficients. We will use the simpler notation $\mathbf{DA}(S)$ for it. We will also encounter $\mathbf{DA}^{\text{eff}}(S) := \mathbf{DA}^{\text{eff,ét}}(S, \mathbb{Q})$, $\mathbf{DM}^{\text{eff}}(S) := \mathbf{DM}^{\text{eff,ét}}(S, \mathbb{Q})$ and $\mathbf{DM}(S) := \mathbf{DM}^{\text{ét}}(S, \mathbb{Q})$.

The conjecture on the existence of the motivic t-structures on $\mathbf{DA}(S)$ are widely open. If one still wants to use the categorical machinery as opposed to the study of algebraic cycles or K-theory, one has to lower one's ambitions and concentrate on the study of subcategories of $\mathbf{DA}(S)$ defined by some geometric conditions. In particular, one can restrict the relative dimensions of the morphisms involved to be less than an integer n. This leads to two closely related theories, homological n-motives $\mathbf{DA}_n(S)$ and cohomological n-motives $\mathbf{DA}^n(S)$ (with similar definitions and notations for other variants). One expects the standard (but not the perverse) motivic t-structure to restrict to these subcategories.

1.1.3 0 and 1-motives over a field

When S is the spectrum of a perfect field k, the structure of the categories $\mathbf{DM}_0^{\text{eff}}(k)$ and $\mathbf{DM}_1^{\text{eff}}(k)$ has been completely described by several authors (Voevodsky, Orgogozo, Barbieri-Viale-Kahn, Ayoub-Barbieri-Viale). The case of 0-motives is very easy and was already fully treated by Voevodsky.

Proposition 1.1.1. *The category $\mathbf{DM}_0^{\text{eff}}(k)$ is equivalent to the derived category of continous $\mathbb{Q}[\mathrm{Gal}(\bar{k}/k)]$-modules.*

To state the results in the case of 1-motives, we need a digression into an older chapter of the theory of mixed motives: Deligne's theory of 1-motives [Del74, §10]. We have already discussed above how Jacobians are in some sense motives of smooth projective curves. As part of his theory of mixed Hodge structures, Deligne set out to describe motives of singular, open curves. He introduced an abelian category $\mathcal{M}_1(k)$ of mixed 1-motives with rational coefficients. An object of this category is a complex $[L \to G]$ with L a lattice and G a semi-abelian variety. In the case of the cohomological 1-motive attached to a curve, the abelian part of G accounts for weight 1 classes in H^1 (in fact, the Jacobian of its smooth projective model), the discrete group L for the weight 0 classes ("caused" by singular points) and the toric part of G for weight 2 classes ("caused" by punctures). We can now describe $\mathbf{DM}_{1,\text{gm}}^{\text{eff}}(k)$.

Theorem 1.1.1 ([Org04])**.** *The category $\mathbf{DM}_{1,\text{gm}}^{\text{eff}}(k)$ is equivalent to the derived category $D^b(\mathcal{M}_1(k))$. Via this equivalence it is equipped with a t-structure.*

The functor $D^b(\mathcal{M}_1(k)) \to \mathbf{DM}^{\mathrm{eff},\text{ét}}_{1,\mathrm{gm}}(k)$ simply associates to a 1-motive $[L \to G]$ the complex of sheaves with transferts $[L^{\mathrm{tr}} \to G^{\mathrm{tr}}] \otimes \mathbb{Q}$ on the category Sm/k of smooth varieties over k.

While motives of 0 and 1-dimensional varieties are not extremely interesting in themselves, it is well known that aspects of the theory of algebraic cycles on higher dimensional varieties are captured by abelian varieties. Divisors on a smooth projective variety X over k are closely related to the Picard abelian variety $\mathrm{Pic}^0(X/k)^{\mathrm{red}}$ and 0-cycles on X are (somewhat less closely) related to the Albanese abelian variety $\mathrm{Alb}(X/k)$. Moreover $\mathrm{Alb}(X/k)$ and $\mathrm{Pic}^0(X/k)^{\mathrm{red}}$ are dual abelian varieties. At the level of motives, this manifests itself as the following result.

Theorem 1.1.2 ([BVK10]). *The inclusion functors*

$$\mathbf{DM}^{\mathrm{eff},\text{ét}}_0(k,\mathbb{Q}) \to \mathbf{DM}^{\mathrm{eff},\text{ét}}(k,\mathbb{Q}) \ \textit{(resp.}\ \ \mathbf{DM}^{\mathrm{eff},\text{ét}}_1(k,\mathbb{Q}) \to \mathbf{DM}^{\mathrm{eff},\text{ét}}(k,\mathbb{Q}))$$

both admit left adjoints $L\pi_0$ (resp. $L\mathrm{Alb}$).

Using the functor $L\mathrm{Alb}$, one can prove a version of Deligne's conjecture on 1-motives attached to the "obviously 1-motivic part" of the cohomology of algebraic variety; see [BVK10] for these applications and others.

Note that both theorems above are proved in [BVK10] with $\mathbb{Z}[1/p]$-coefficients (where p is the exponential characteristic of k), however we will not look at such a refinement in this thesis.

A word of warning: even over a field, the category $\mathbf{DM}^{\mathrm{eff}}_2(k)$ of 2-motives is not understood at all. One has no candidate for "geometric" models of motives of surfaces, and indeed Mumford's theorem on 0-cycles of surfaces show that the naive expectation of parametrising higher codimensional cycles up to rational equivalence by algebraic varieties is impossible. There is another definition of an abelian subcategory of 2-motives in $\mathbf{DM}^{\mathrm{eff}}(k)$ (see [Ayo11]), but showing that it is related to $\mathbf{DM}^{\mathrm{eff}}_2(k)$ seems difficult. In particular the Beilinson-Soule vanishing conjecture for $\mathbb{Q}(2)$ and Bloch's conjecture on 0-cycles, which would follow from the existence of $\mathbf{MM}_2(k)$, are completely open. It would be interesting to know if those two conjectures together suffice to construct the t-structure for 2-motives.

1.1.4 Motives of commutative algebraic groups

Let k be a perfect field and G/k be a commutative algebraic group. The cohomology groups of G for all standard cohomology theories exhibit a simple structure: we have $H^*(G) := \Lambda^* H^1(G)$ as an Hopf algebra. The paper [AEWH13] provides a lift of this result to motives via a Chow-Künneth decomposition of the motive $M_k^{\mathrm{eff,tr}}(G)$ in $\mathbf{DM}^{\mathrm{eff}}(k)$: there is an isomorphism

$$M_k(G) \xrightarrow{\sim} \left(\bigoplus_{n\geq 0}^{\mathrm{kd}(G/k)} \mathsf{Sym}^n G^{\mathrm{tr}}_{\mathbb{Q}} \right) \otimes M_k^{\mathrm{eff,tr}}(\pi_0(G/k)) \ . \qquad (*_k)$$

In the above, $G^{\mathrm{tr}}_{\mathbb{Q}}$ is the rational 1-motive of G (effective, with transfers), i.e., the motive induced by the étale sheaf with transfers represented by $G \otimes \mathbb{Q}$, the motive $M_k^{\mathrm{eff,tr}}(\pi_0(G/k))$ is a fibrewise Artin motive related to the connected components of fibres of G and $\mathrm{kd}(G/k)$ is a non-negative integer. This decomposition is natural in G and respects the structure of Hopf algebra objects.

1.2 The results of this thesis

1.2.1 Motives of smooth commutative group schemes

The results in this section were obtained in joint work with Giuseppe Ancona and Annette Huber. Let S be a noetherian scheme of finite dimension and G a smooth commutative S-group scheme of finite type. We establish the following canonical Künneth decomposition of the motive $M_S(G)$ associated to G in the triangulated category of motives over S with rational coefficients (see Theorems 2.3.3 and 2.3.7):

$$M_S(G) \xrightarrow{\sim} \left(\bigoplus_{n\geq 0}^{\mathrm{kd}(G/S)} \mathsf{Sym}^n \Sigma^\infty G_{\mathbb{Q}} \right) \otimes M_S(\pi_0(G/S)) \ . \qquad (*_S)$$

In the above, $\Sigma^\infty G_{\mathbb{Q}}$ is the rational 1-motive of G, i.e., the motive induced by the suspension T-spectrum on the étale sheaf represented by $G \otimes \mathbb{Q}$, the motive $M_S(\pi_0(G/S))$ is a fibrewise Artin motive related to the connected components of fibres of G and $\mathrm{kd}(G/S)$ is a non-negative integer. This decomposition is natural in G, it respects the structure of Hopf algebra objects and it is compatible with pullback for morphisms $f : T \to S$.

This level of generality makes it possible to treat examples like degenerating families of abelian or semi-abelian varieties, in particular Néron models or the universal family over toroidal compactifications of mixed Shimura varieties of Hodge type. Note also that there are no assumptions on regularity or the residual characteristics of S. Moreover, we show that this Künneth decomposition also holds when G is a commutative group in the category of smooth algebraic spaces over S. However, the argument uses rational coefficients in many places. One does not expect such a direct sum decomposition with integral coefficients, even when S is the spectrum of a field.

This result has several interesting consequences. First, the motive $\Sigma^\infty G_{\mathbb{Q}}$ is geometric and the motives $M_S(G)$ and $\Sigma^\infty G_{\mathbb{Q}}$ are of finite Kimura dimension in the sense of [Kim05]. Second, the Chow groups of G with rational coefficients decompose into a finite sum of eigenspaces with respect to the multiplication by n (Theorem 2.3.10); this generalizes the result of Beauville for abelian varieties [Bea86]. Third, this can be used to construct the motivic polylog on $G \smallsetminus \{e_G\}$ for all S and G as above (with e_G the zero section); for details see the paper [HK15].

The motive $\Sigma^\infty G_{\mathbb{Q}}$ is a prime example of a relative homological 1-motive, i.e., a motive in the triangulated subcategory generated by the relative homology of curves over S. In fact, the results of Section 2.5 below suggest that $\Sigma^\infty G_{\mathbb{Q}}[-1]$ is in the heart of the conjectural standard motivic t-structure on homological 1-motives. These results will be established in the second part of this thesis.

Some special cases of $(*_S)$ were already known. The case of abelian schemes over a regular base was treated in the setting of Chow motives by Künnemann in [Kün94], building on the theorem of Deninger and Murre mentioned above. Starting from this, the case where S is the spectrum of a perfect field and general G was treated by the first two authors together with Enright-Ward in [AEWH13], as mentioned above.

The idea of the proof is simple: once the morphism $(*_S)$ is written down, we can check that it is an isomorphism after pullback to geometric points of S, where we are back in the perfect field case and can use $(*_k)$.

However, a number of difficulties arise. Firstly, there are technical advantages to working in the setting of motives without transfers (see Remark 2.2.4) and it is in this setting that we define the morphism $(*_S)$. So we will have to compare this definition with the approach of [AEWH13] which uses Voevodsky's category of motives with transfers. Secondly, the sheaf represented by G is not cofibrant for the $\mathbb{A}^1$-model structure; hence, it is not obvious how to compute the left derived pullback $f^*\Sigma^\infty G_{\mathbb{Q}}$. This is overcome by an explicit resolution defined by rational homotopy theory. Finally, the crucial reduction that allows one to check that a morphism is an isomorphism after restriction to geometric fibres (Lemma 2.4.2) is only available in the stable case.

All these technical difficulties give rise to results that have their own interest. In particular we show that the sheaf represented by $G \otimes \mathbb{Q}$ admits transfers (Theorem 2.2.8).

1.2.2 Relative 1-motives

Let S be a finite dimensional noetherian scheme. In Section 3.1, we introduce natural definitions of categories $\mathbf{DA}_n(S)$ of homological n-motives (resp. $\mathbf{DA}^n(S)$ of cohomological n-motives) which are full subcategories of $\mathbf{DA}(S)$ generated as triangulated categories with small sums by homological (resp. cohomological) motives of smooth (resp. proper) S-schemes of relative dimension less than n (Definition 3.1.1). We then study their permanence properties under the six operations (Propositions 3.1.10 to 3.1.18) and prove that the homological and cohomological variants are closely related (Proposition 3.1.28). Unfortunately, for $n \geq 2$, we cannot say much more about these categories; the cases $n = 0, 1$ are more tractable and are explored in the rest of this chapter.

In Section 3.2, we come back to the study of motives attached to smooth commutative group schemes over S and prove that they live in $\mathbf{DA}_1(S)$ (Proposition 3.2.14). The relationship between 1-motives and motives of the form $\Sigma^\infty G_{\mathbb{Q}}$ is the key to everything that follows. We also study motives attached to Deligne 1-motives. Finally, we introduce a motive attached to what we call the Picard complex $P(X/S)$ of a morphism of schemes $f : X \to S$, a complex of sheaves which packages together the information about the relative connected components of f and the Picard scheme of X/S; under certain hypotheses, this yields a motive in $\mathbf{DA}_1(S)$ (Corollary 3.2.34).

In Section 3.3, we introduce and study the right adjoint $\omega^1 : \mathbf{DA}^{\mathrm{coh}}(S) \to \mathbf{DA}^1(S)$ to the embedding of cohomological 1-motives into cohomological motives. We first establish a number of relatively formal results involving its commutation properties with the six operations (Proposition 3.3.3. The main result is then that ω^1 preserves constructibility (Theorem 3.3.12). This relies on combining techniques from [AZ12] with a computation of $\omega^1(f_*\mathbb{Q}_X)$ in a favorable situation: the precise statement involves the motive of the Picard complex from the previous section.

In Section 3.4, we finally introduce a candidate for the motivic t-structure on $\mathbf{DA}_1(S)$ and $\mathbf{DA}^1(S)$, using the formalism of generated t-structures. A number of equivalent generating families can be used for this purpose (see Definition 3.4.2). We prove some basic exactness properties for the six operations. The main result we show is that motives attached to Deligne 1-motives lie in the heart $\mathbf{MM}_1(S)$, and that more precisely the category $\mathcal{M}_1(S)$ embeds fully faithfully into $\mathbf{MM}_1(S)$ for S regular. Finally, we state some conjectures about the t-structure which should be accessible with small extensions of the methods of this thesis.

Appendix 3.A discusses Deligne 1-motives over a general base and proves a continuity property for them (Proposition 3.A.9). Appendix 3.B gathers some computations of motivic cohomology groups for $\mathbb{Q}(0)$ and $\mathbb{Q}(1)$ which are used at several places in the text.

Background, conventions and notations

We collect here several conventions and pieces of notations which will be used throughout this thesis.

Homological algebra in abelian and triangulated categories

When discussing complexes in abelian categories and t-structures on triangulated categories, we consistently use homological indexing conventions.

Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangulated functor between triangulated categories with t-structures. We say that F is t-positive (resp. t-negative) if $F(\mathcal{T}_{\geq 0}) \subset \mathcal{T}'_{\geq 0}$ (resp. $F(\mathcal{T}_{\leq 0}) \subset \mathcal{T}'_{\leq 0}$).

Let $\mathcal{T}$ be a triangulated category, and $\mathcal{G}$ a family of objects of $\mathcal{T}$. We introduce a number of subcategories of $\mathcal{T}$ generated in various ways by $\mathcal{G}$. We denote by $\langle \mathcal{G} \rangle$ (resp. $\langle \mathcal{G} \rangle_+$, $\langle \mathcal{G} \rangle_-$) the smallest triangulated subcategory of $\mathcal{T}$ stable by direct factors (resp. the smallest subcategory stable by extensions, positive shifts and direct factors, the smallest subcategory stable by extensions, negative shifts and direct factors) containing $\mathcal{G}$. Assume now that $\mathcal{T}$ admits small sums. We denote by $\ll \mathcal{G} \gg$ (resp. $\ll \mathcal{G} \gg_+$, $\ll \mathcal{G} \gg_-$) the smallest triangulated subcategory of $\mathcal{T}$ (resp. the smallest subcategory stable by extensions, small sums and $[+1]$, the smallest subcategory stable by extensions, small sums and $[-1]$) containing $\mathcal{G}$. Note that $\langle \mathcal{G} \rangle \subset \ll \mathcal{G} \gg$ by [Ayo07a, Lemme 2.1.17].

We refer informally to $\mathcal{G}$ as the *generating family* of any of the subcategories above and to objects of $\mathcal{G}$ as *generators*. In each case, these categories can be defined by a (possibly transfinite in the $\ll - \gg$ cases) induction: we start with the full subcategory with objects $\mathcal{G}[\mathbb{Z}]$; to pass to a successor ordinal we introduce, depending on the case, cones of all morphisms and direct factors of all objects, just the cones and direct factors, just the cocones and direct factors, the cones, direct factors and small sums, ...; finally, to pass to a limit ordinal we take the union over all previous subcategories. These subcategories do not change if one replaces $\mathcal{T}$ by a triangulated subcategory containing $\mathcal{G}$ (and stable under small sums for the $\ll - \gg$ variants), so that we will in general not need to specify the ambient triangulated category, which we omit from the notation.

Schemes and group schemes

Unless specified, all schemes are noetherian and finite dimensional, all morphisms of schemes are of finite type, and all smooth morphisms (resp. étale morphisms) are separated of finite type (resp. separated quasi-finite). The notation Sm/S (resp. Sch/S) denotes the category of all smooth S-schemes (resp. all separated finite type S-schemes), usually considered as a site with the étale topology.

We say that a scheme S admits the resolution of singularities by alterations if for any separated S-scheme X of finite type and any nowhere dense closed subset $Z \subset X$, there is a projective Galois alteration $g : X' \to X$ with X' regular and such that $g^{-1}(Z)$ is a normal crossing divisor. Recall that by [dJ97, Corollary 5.15], any separated finite type scheme over a noetherian excellent base of dimension ≤ 2 admits the resolution of singularities by alterations.

Let us recall basic terminology and facts about exact sequences of group schemes. Let (C) : $0 \to G' \xrightarrow{i} G \xrightarrow{p} G'' \to 0$ be a sequence of commutative group schemes over a scheme S. We say that (C) is exact if it induces an exact sequence of fppf sheaves on Sch/S. If (C) is exact, then G' is the scheme-theoretic kernel of p and p is a surjective morphism of schemes. In the other direction, if p is an fppf morphism and G' is its scheme-theoretic kernel, then (C) is exact. Moreover, if the group schemes involved are smooth, then one obtains an equivalent definition (and results) by replacing the fppf topology with the étale topology.

Triangulated categories of motives

We work in most cases in the context of the stable homotopical 2-functor $\mathbf{DA}^{\text{ét}}(-, \mathbb{Q})$ considered in [Ayo14a, §3].

Since we only consider the étale topology and rational coefficients, we simplify the notation and write $\mathbf{DA}(S)$ for $\mathbf{DA}^{\text{ét}}(S, \mathbb{Q})$. The category $\mathbf{DA}(S)$ is equivalent to several other constructions of motives with rational coefficients, e.g. Beilinson motives as in [CDb]: see [CDb, §16] for various comparison theorems. Let us briefly recall the construction of $\mathbf{DA}(S)$.

Definition 1.2.1. Let X be a smooth S-scheme. We write $\mathbb{Q}(X) \in \mathbf{Sh}_{\text{ét}}(\mathsf{Sm})$ for the sheafification of the presheaf

$$\mathbb{Q}\,\mathrm{Mor}_S(-, X)\ ,$$

which associates to each smooth S-scheme Y the $\mathbb{Q}$-vector space with basis the set of S-morphisms from Y to X.

We denote by $D(\mathsf{Sm}/S)$ the derived category of étale sheaves of $\mathbb{Q}$-vector spaces on the site Sm/S. Write T for the sheaf-theoretic cokernel of the unit section $\mathbb{Q}(S) \to \mathbb{Q}(\mathbb{G}_{m,S})$; we view T as a convenient model for the Tate motive. The category $\mathbf{DA}(S)$ is defined to be the homotopy category of a certain stable $\mathbb{A}^1$-local model category of symmetric T-spectra of complexes in the abelian category $\mathbf{Sh}_{\text{ét}}(\mathsf{Sm})$ of étale sheaves in $\mathbb{Q}$-vector spaces on Sm/S. For simplicity, we refer to an object in this model category as a *motivic spectrum.* The analogous construction without spectra leads to the category $\mathbf{DA}^{\text{eff}}(S)$ of *effective triangulated motives*, which is the $\mathbb{A}^1$-localization of $D(\mathsf{Sm}/S)$. For simplicity, we refer to an object in this category as a *motivic complex.* At several points, we have parallel statements in the effective and stable categories. They are indicated by an (eff) in the notation.

Recall that to compute derived functors in this setting, one chooses a specific model category structure on (spectra of) complexes of sheaves presenting the categories of motives such that the functors of interest are Quillen functors. For the result of this thesis, we use the projective model structures (adapted from presheaves to sheaves) of [Ayo14b, Section 3].

There is a sequence of functors (see [CD09, 5.3.23.2] for more details on the last one, the *infinite T-suspension functor*)

$$\mathbf{Sh}_{\text{ét}}(S) \to D(\mathbf{Sh}_{\text{ét}}(\mathsf{Sm})) \to \mathbf{DA}^{\text{eff}}(S) \stackrel{L\Sigma^\infty}{\to} \mathbf{DA}(S). \tag{1.2.2.1}$$

Definition 1.2.2. Let X be a smooth S-scheme.

1. The *(relative, homological) effective motive* $M_S^{\text{eff}}(X)$ is defined as the image of $\mathbb{Q}(X)$ in $\mathbf{DA}^{\text{eff}}(S)$.

2. The motive $M_S(X)$ is defined as $L\Sigma^\infty M_S^{\text{eff}}(X)$ in $\mathbf{DA}(S)$.

Motivic complexes (respectively, spectra) of the form $\mathbb{Q}(X)$ (respectively, $\Sigma^\infty\mathbb{Q}(X)$) are called *representable.*

The category $\mathbf{DA}_{\text{gm}}(S)$ of *constructible* or *geometric* motives over S is defined as the thick triangulated subcategory of $\mathbf{DA}(S)$ generated by the motives of the form $M_S(X)(i)$ for any smooth S-scheme X and any integer $i \in \mathbb{Z}$.

Remark 1.2.3. The projective model structures that we are using have relatively few cofibrant objects - as they should, since we want to use them to compute left derived functors - but crucially representable objects are cofibrant (by definition for the descent model structure of [CD09, 5.1.11], and as they have the left lifting property with respect to surjective quasi-isomorphisms for the projective one).

Remark 1.2.4. A motive is constructible if and only if it is a *compact* object of the triangulated category $\mathbf{DA}(S)$ (see [Ayo14b, Proposition 8.3]).

By [Ayo07a], the functor $\mathbf{DA}(-)$ admits the functoriality of the Grothendieck six operations. In particular, for any quasi-projective morphism $f : S \to T$, Ayoub constructs adjoint pairs

$$f^* : \mathbf{DA}(T) \leftrightarrows \mathbf{DA}(S) : f_*$$

$$f_! : \mathbf{DA}(S) \leftrightarrows \mathbf{DA}(T) : f^!$$

and when f is smooth

$$f_\sharp : \mathbf{DA}(S) \leftrightarrows \mathbf{DA}(T) : f^*.$$

There is a morphism of functors $f_! \to f_*$, which is an isomorphism for f projective.

Note that for those operations, as well as for the pullbacks and pushforwards functors on derived categories of sheaves on $\mathsf{Sm}/-$, the notation f^*, f_*, ... stands for the triangulated or derived functor. When we want to use the underived functor, we underline the functor: $\underline{f}^*, \underline{f}_*, \ldots$

In the definitions of the Grothendieck operations, one can relax the condition f quasi-projective in the following ways.

(i) As observed in [Ayo, Appendice 1.A], one can define f^* and f_* for any morphism f (without any finiteness hypothesis), and prove for instance that proper base change [Ayo14a, Proposition 3.5], the $\mathrm{Ex}^*_\sharp$ isomorphism [Ayo14a, Proposition 3.6] and "regular base change" [Ayo, Corollaire 1.A.4] still hold.

(ii) As observed in [CDb, Theorem 2.2.14], one can define the exceptional functors $f_!$ and $f^!$ for any f separated of finite type, and prove that all the properties in [Ayo07a] still hold (with in particular $f_! \simeq f_*$ for any f proper).

(iii) In fact, one can define the four operations for all morphisms locally of finite type between quasi-compact quasi-separated schemes, as is explained in [Hoy14, Appendix C].

In this paper, we use the extensions (i)-(ii) but not (iii).

The six operations for $\mathbf{DA}(-)$ satisfy a large number of properties and compatibilities (see [Ayo14a, Proposition 3.2], [Ayo07a, Scholie 1.4.2]). Here are some properties of the monoidal structure.

Lemma 1.2.5. *Let $f : S \to T$ be a morphism of noetherian schemes of finite dimension.*

1. *All functors in the sequence (1.2.2.1) are monoidal. In particular, if X, Y are objects in Sm/S, then*

$$\mathbb{Q}(X) \otimes \mathbb{Q}(Y) \simeq \mathbb{Q}(X \times Y) \ ,$$
$$M_S^{(\mathrm{eff})}(X) \otimes M_S^{(\mathrm{eff})}(Y) \simeq M_S^{(\mathrm{eff})}(X \times Y) \ .$$

2. *For $X \in \mathsf{Sm}/S$, we have canonical isomorphisms:*

$$f^*\mathbb{Q}(X) \simeq \mathbb{Q}(X_T) \text{ and } f^*M_S^{(\mathrm{eff})}(X) \simeq M_T^{(\mathrm{eff})}(X_T) \ .$$

3. *There is a canonical 2-isomorphism $L\Sigma^\infty f^* \simeq f^* L\Sigma^\infty$.*

4. *The pullback functor f^* is monoidal.*

For other results which come up repeatedly in this thesis, we introduce the following shorthand. Let

$$\begin{tikzcd} Z \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ W \arrow[r, "g"'] & Y \end{tikzcd}$$

be a cartesian square.

- By the $\mathrm{Ex}^*_\sharp$ isomorphism (resp. the $\mathrm{Ex}^!_*$ isomorphism, the $\mathrm{Ex}^*_!$ isomorphism), we mean the natural isomorphism $\tilde{f}_\sharp \tilde{g}^* \xrightarrow{\sim} g^* f_\sharp$ for f smooth (resp. the natural isomorphism $\tilde{f}_* \tilde{g}^! \xrightarrow{\sim} g^! f_*$, the natural isomorphism $g^* f_! \xrightarrow{\sim} \tilde{f}_! \tilde{g}^!$).
- By "smooth base change", we mean the natural isomorphism $\tilde{f}_* \tilde{g}^* \xrightarrow{\sim} g^* f_*$ for g smooth.
- By "proper base change", we mean the natural isomorphism $g^* f_! \xrightarrow{\sim} \tilde{f}_! \tilde{g}^*$ for f proper.

- Let $i : Z \to X$ be a closed immersion and $j : U \to X$ be the complementary open immersion. When we write "by localisation", we mean the implicit use of the distinguished triangle of functors
$$j_\sharp j^* \to \mathrm{id} \to i_* i^* \xrightarrow{+} .$$
Dually, when we write "by colocalisation", we mean the implicit use of the distinguished triangle of functors
$$i_! i^! \to \mathrm{id} \to j_* j^* \xrightarrow{+} .$$

- By "relative purity", we mean the fact that for any smooth morphism $f : S \to T$ of relative dimension d, there are isomorphism of functors $f_! \simeq f_\sharp(d)[2d]$ and $f^! \simeq f^*(-d)[-2d]$.

- By "separation" or "the separation property for $\mathbf{DA}$", we mean the fact that for any surjective morphism of finite type (resp. any finite surjective radicial morphism) $f : S \to T$, the functor $f^* : \mathbf{DA}(T) \to \mathbf{DA}(S)$ is conservative (resp. an equivalence of categories) [Ayo14a, Theoreme 3.9].

- By "absolute purity", we mean the fact that for any closed immersion $i : S \to T$ of codimension d with S, T regular schemes, we have $i^! \mathbb{Q}_T \simeq \mathbb{Q}_S(-d)[-2d]$ ([Ayo14a, Corollaire 7.5] and [Ayo14a, Remarque 11.2]).

We need to consider motives of algebraic spaces. The definition follows the method of Choudhury [Cho12], who considered the case of stacks over a field. We only develop the minimal amount of results necessary.

Let Y be an algebraic space presented by a scheme X and an étale equivalence relation R, i.e., Y is the cokernel of the diagram of sheaves of sets over the site Sch/S with the étale topology
$$R \rightrightarrows X .$$

Definition 1.2.6. Let Y be a smooth algebraic space. Let $\mathbb{Q}(Y)$ be the sheaf associated to the presheaf $\mathbb{Q}Y_{|\mathsf{Sm}/S}(\cdot)$. We define the relative motive $M_S^{\mathrm{eff}}(Y) \in \mathbf{DA}^{\mathrm{eff}}(S)$ as the image of the sheaf $\mathbb{Q}(Y)$ and $M_S(Y) \in \mathbf{DA}(S)$ as $L\Sigma^\infty M_S^{\mathrm{eff}}(Y)$.

If Y is a scheme, this agrees with the definition given before.

Proposition 1.2.7. *Let $f : T \to S$ be a morphism and Y/S be a smooth algebraic space. Then there are canonical isomorphisms*
$$f^* M_S^{(\mathrm{eff})}(Y) \simeq M_T^{(\mathrm{eff})}(Y_T).$$

Proof. By Lemma 1.2.5(3) it is enough to settle the effective statement.

We follow the method of [Cho12, Cor. 2.14]. Let (X, R) be a presentation of Y and $X_\bullet$ be the Cech nerve of the cover $X \to Y$. By general principles, the covering map
$$X_\bullet \to Y$$
is a weak equivalence of simplicial sheaves. Hence it induces a quasi-isomorphism of complexes of sheaves of $\mathbb{Q}$-vectors spaces
$$\mathbb{Q}(X_\bullet) \to \mathbb{Q}(Y) .$$
We are going to show that X_n is a smooth S-scheme. Assuming this, we compute $f^* M_S(X)$ as
$$f^* \mathbb{Q}(X_\bullet) = \mathbb{Q}((X_\bullet)_T) .$$
As $(X_\bullet)_T$ is the Cech nerve of the cover $X_T \to Y_T$,the $\mathbb{Q}((X_\bullet)_T)$ is quasi-isomorphic to $\mathbb{Q}(Y_T)$.

By definition, X_n is the $(n+1)$-fold fibre product of sheaves of sets
$$X_n = X \times_Y \times X \cdots \times_Y X = (X \times_Y X) \times_X \cdots \times_X (X \times_Y X) .$$
Since $X \times_Y X \simeq R$ we see that X_n is a smooth X-scheme. As X is smooth, this makes X_n a smooth S-scheme. $\square$

Triangulated categories with and without transfers

Let S be a noetherian finite dimensional scheme. We will occasionally need to work with categories of motives with transfers. We recall the necessary definitions and results.

Definition 1.2.8. 1. Let $\mathsf{Sm}^{\mathrm{tr}}$ be the category of smooth correspondences over S. Its objects are the objects of Sm and morphisms are finite $\mathbb{Q}$-correspondences in the sense of [CD09, Definition 9.1.2 - Definition 9.1.8]. (This category has also other notations in the literature as $\mathsf{Sm}^{\mathrm{cor}}$ in [CD09] and SmCor in [Voe00]).

2. Let $\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}^{\mathrm{tr}})$ be the category of additive presheaves of $\mathbb{Q}$-vector spaces on $\mathsf{Sm}^{\mathrm{tr}}$ whose restriction to Sm is an étale sheaf. We call these objects *étale sheaves with transfers.*

3. Let $X \in \mathsf{Sm}$. Then $\mathbb{Q}(X)^{\mathrm{tr}}$ is defined as the representable presheaf with transfers $c_S(\cdot, X)$ (which is in fact an étale sheaf).

The category $\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}^{\mathrm{tr}})$ is monoidal. The tensor product is characterized by the formula

$$\mathbb{Q}(X)^{\mathrm{tr}} \otimes \mathbb{Q}(Y)^{\mathrm{tr}} = \mathbb{Q}(X \times Y)^{\mathrm{tr}}$$

for $X, Y \in \mathsf{Sm}$.

By [CD13, Corollary 2.1.12] , there is an adjoint pair of functors

$$\underline{a}_{\mathrm{tr}} : \mathbf{Sh}_{\text{ét}}(\mathsf{Sm}) \rightleftarrows \mathbf{Sh}_{\text{ét}}(\mathsf{Sm}^{\mathrm{tr}}) : \underline{o}^{\mathrm{tr}} \ ,$$

where $\underline{o}^{\mathrm{tr}}$ is simply the restriction from sheaves with transfers to sheaves without transfer, and $\underline{a}_{\mathrm{tr}}$ verifies the following properties.

Lemma 1.2.9. *Let $\underline{a}_{\mathrm{tr}}$ and $\underline{o}^{\mathrm{tr}}$ the functors defined above. Then, for all S-smooth schemes $p : X \to S$ and all sheaves with transfers F, the following statements hold:*

1. $\underline{a}_{\mathrm{tr}}\mathbb{Q}(X) = \mathbb{Q}(X)^{\mathrm{tr}}$,
2. *the functor $\underline{a}_{\mathrm{tr}}$ is monoidal.*

Proof. By [CD09, Corollary 10.3.11], the functor $\underline{o}^{\mathrm{tr}}$ induces an adjunction of abelian $\mathcal{P}$-premotivic categories [CD09, Definition 1.4.6] (with $\mathcal{P} = \mathsf{Sm}$). This implies both properties of the statement. Indeed, by definition, $\underline{a}_{\mathrm{tr}}$ is in particular a morphism of monoidal $\mathcal{P}$-fibered categories [CD09, Definition 1.2.7] (which implies 2) and commutes with the functors $p_{\#}$ (which implies 1). $\square$

There is an alternative description of $\underline{a}_{\mathrm{tr}}$ via the category of qfh-sheaves. Let Sch be the category of schemes of finite type over S. The inclusion of categories induces a morphism of sites $p : \mathsf{Sch}_{\text{ét}} \to \mathsf{Sm}_{\text{ét}}$ and a pair of adjoint functors

$$p^* : \mathbf{Sh}_{\text{ét}}(\mathsf{Sm}) \rightleftarrows \mathbf{Sh}_{\text{ét}}(\mathsf{Sch}) : p_* \ ,$$

where p_* is simply the restriction. We also write more suggestively $F|_{\mathsf{Sm}} = p_*F$. Recall the qfh-topology on Sch from [SV00, Section 4.1] . Roughly, it is the topology generated by open covers and finite surjective morphisms. As the qfh topology refines the étale topology, there is a morphism of sites $r : \mathsf{Sch}_{\mathrm{qfh}} \to \mathsf{Sch}_{\text{ét}}$ and a pair of adjoint functors

$$r^* : \mathbf{Sh}_{\text{ét}}(\mathsf{Sch}) \rightleftarrows \mathbf{Sh}_{\mathrm{qfh}}(\mathsf{Sch}) : r_*.$$

We also write more suggestively $F_{\mathrm{qfh}} = r^*F$. In particular, $\mathbb{Q}(X)_{\mathrm{qfh}}$ is the sheafification of the presheaf $\mathbb{Q}(X)$ on Sch. As usual we denote by $\mathbb{Q}(X)^{\mathrm{tr}} \in \mathbf{Sh}_{\text{ét}}(\mathsf{Sm}^{\mathrm{tr}})$ the representable sheaf with transfers defined by X.

The following result is due to Suslin and Voevodsky [SV00, Proposition 4.2.7, Theorem 4.2.12].

Theorem 1.2.10. *For $X \in \mathsf{Sm}$, the canonical map $\mathbb{Q}(X) \to \underline{o}^{\mathrm{tr}}\mathbb{Q}(X)^{\mathrm{tr}}$ factors through the qfh sheafification and induces an isomorphism:*

$$\mathbb{Q}(X)_{\mathrm{qfh}}|_{\mathsf{Sm}} \xrightarrow{\sim} \underline{o}^{\mathrm{tr}}\mathbb{Q}(X)^{\mathrm{tr}} \ .$$

This implies that any qfh-sheaf has canonical transfers; see [CD09, Proposition 10.5.8].

A priori, the functor $\underline{a}_{\mathrm{tr}}$ is only right exact. (The above theorem suggests that $\underline{a}_{\mathrm{tr}}$ should be equal to $p_* r_* r^* p^*$, the source of non-exactness being r_*.) It is not clear whether it is also left exact. However, we have the following criterion.

Lemma 1.2.11. *Assume S normal. Let F be an étale sheaf of $\mathbb{Q}$-vector spaces on Sch and*

$$C^* \to F$$

a resolution in the same category such every C^i is of the form $\mathbb{Q}(X_i)$ for some smooth X_i. Then the following holds.

(i) *The complex*

$$\underline{a}_{\mathrm{tr}}(C^*|_{\mathsf{Sm}}) \to \underline{a}_{\mathrm{tr}}(F|_{\mathsf{Sm}}) \to 0$$

of objects in $\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}^{\mathrm{tr}})$ is exact.

(ii) *There is a canonical isomorphism*

$$(F_{\mathrm{qfh}})|_{\mathsf{Sm}} \xrightarrow{\sim} \underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(F|_{\mathsf{Sm}}).$$

and this isomorphism identifies the unit map $F|_{\mathsf{Sm}} \to \underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(F|_{\mathsf{Sm}})$ and the unit map $F|_{\mathsf{Sm}} \to (F_{\mathrm{qfh}})|_{\mathsf{Sm}}$.

Proof. The complex $C^*_{\mathrm{qfh}} \to F_{\mathrm{qfh}} \to 0$ is an exact complex of qfh sheaves on Sch. We claim that its restriction to Sm is an exact complex of étale sheaves. Since S (and hence every scheme in Sm) is normal and we work with sheaves of $\mathbb{Q}$-vector spaces, this follows from a trace argument (see [Voe96, Theorem 3.4.1]).

Because of the assumption on the C^i, Theorem 1.2.10 implies that the natural map $C^*|_{\mathsf{Sm}} \to \underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(C^*|_{\mathsf{Sm}})$ factors through $C^*_{\mathrm{qfh}}|_{\mathsf{Sm}}$ and that we get an isomorphism of complexes in $\mathbf{Sh}_{\text{ét}}(\mathsf{Sm})$:

$$\delta : C^*_{\mathrm{qfh}}|_{\mathsf{Sm}} \xrightarrow{\sim} \underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(C^*|_{\mathsf{Sm}}).$$

This implies that the complex $\underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(C^*|_{\mathsf{Sm}})$ is exact. Since $\underline{a}_{\mathrm{tr}}$ is right exact and $\underline{o}^{\mathrm{tr}}$ is exact, we get that the complex

$$\underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(C^{-1}|_{\mathsf{Sm}}) \to \underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(C^{0}|_{\mathsf{Sm}}) \to \underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(F|_{\mathsf{Sm}}) \to 0$$

is also exact. Put together, this implies that $\underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(C^*|_{\mathsf{Sm}}) \to \underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}(F|_{\mathsf{Sm}}) \to 0$ is exact. Since $\underline{o}^{\mathrm{tr}}$ is exact and faithful, this proves point (i). Now the isomorphism δ induces an isomorphism on H^0 which proves the first part of (ii). The second part is formal from the definition of the map. $\square$

We can now introduce the triangulated categories of motives with transfers $\mathbf{DM}^{\mathrm{eff}}(S)$ and $\mathbf{DM}(S)$ which are defined in a completely parallel manner to $\mathbf{DA}^{\mathrm{eff}}(S)$ and $\mathbf{DA}(S)$ by replacing sheaves with sheaves with transfers. They also occur as homotopy categories of model structures, for which we do not go into details and refer to [CD09, Definition 11.1.1].

We have a sequence of functors for motives with transfers

$$\mathbf{Sh}^{\mathrm{tr}}_{\text{ét}}(S) \to D(\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}^{\mathrm{tr}})) \to \mathbf{DM}^{\mathrm{eff}}(S) \overset{L\Sigma^\infty_{\mathrm{tr}}}{\to} \mathbf{DM}(S)\ . \tag{1.2.2.2}$$

The functors $\underline{a}_{\mathrm{tr}}$ and $\underline{o}^{\mathrm{tr}}$ between abelian categories of sheaves induce Quillen adjunctions at the level of the model categories on motivic complexes and spectra (see [CD13, 2.2.6] for the case of the descent model structures):

$$a_{\mathrm{tr}} : \mathbf{DA}^{\mathrm{eff}}(S) \rightleftarrows \mathbf{DM}^{\mathrm{eff}}(S) : o^{\mathrm{tr}}\ .$$

$$a_{\mathrm{tr}} : \mathbf{DA}(S) \rightleftarrows \mathbf{DM}(S) : o^{\mathrm{tr}}\ .$$

The notation reflects:

Lemma 1.2.12. *The functor $\underline{a}_{\mathrm{tr}}$ on motivic complexes preserves $\mathbb{A}^1$-equivalences.*

Proof. The proof of [Ayoarb, Lemme 2.111] also works in our setting. □

Theorem 1.2.13. *[CD09, Theorem 16.2.18] and [CD09, Theorem 16.1.4] Let S be noetherian, finite dimensional, excellent and geometrically unibranch. Then the functor*

$$a_{\mathrm{tr}} : \mathbf{DA}(S) \rightleftarrows \mathbf{DM}(S) : o^{\mathrm{tr}} \ .$$

is an equivalence of categories.

Remark 1.2.14.

1. There is an integral version of this in [Ayo14b, Théorème B.1] (there the functor a_{tr} is denoted La_{tr}.)
2. In the effective case, the optimal comparison result is not known. For the state of the art in the effective case, see [Ayoarb, Annexe B.] (for a base of characteristic 0) and [Vez14, Theorem 3.19] (a weaker version over a perfect base).

Definition 1.2.15. Let X be a smooth S-scheme.

1. The *(relative, homological) effective motive with transfers* $M_S^{\mathrm{eff}}(X)^{\mathrm{tr}}$ is defined as the image of $\mathbb{Q}(X)^{\mathrm{tr}}$ in $\mathbf{DM}^{\mathrm{eff}}(S)$.
2. The *motive with transfers* $M_S(X)^{\mathrm{tr}}$ is defined as $L\Sigma^{\infty}_{\mathrm{tr}} M_S^{\mathrm{eff}}(X)^{\mathrm{tr}}$ in $\mathbf{DM}(S)$.

We collect some useful computational properties:

Lemma 1.2.16. *The following statements hold:*

1. *All functors in the sequence (1.2.2.2) are monoidal.*
2. *There is a functorial 2-isomorphism $a_{\mathrm{tr}}\Sigma^{\infty} \simeq \Sigma^{\infty}_{\mathrm{tr}} a_{\mathrm{tr}}$.*
3. *For all $X \in \mathsf{Sm}$, we have $a_{\mathrm{tr}} M_S(X) \simeq M_S(X)^{\mathrm{tr}}$.*
4. *In both the effective and non-effective case, the functor a_{tr} is monoidal.*

Proof. Assertion (1) follows from [CD09, Proposition 10.1.2] and the general machinery of monoidal $\mathcal{P}$-fibered categories of *loc. cit.*, together with the fact that the tensor product of sheaves with transfers with $\mathbb{Q}$-coefficients is exact.

Assertion (2) is contained in [CD09, Corollary 10.3.11] and [CD09, 5.3.28].

For Assertion (3), the motivic spectrum $\Sigma^{\infty}\mathbb{Q}(X)$ is cofibrant for the model structure on motivic spectra by 1.2.3. Moreover the functor $\underline{a}_{\mathrm{tr}}$ on motivic spectra is left Quillen (this is also in [CD09, Corollary 10.3.11] and [CD09, 5.3.28]) and commutes with suspensions by (2). To conclude we apply Lemma 1.2.9 (1).

Assertion (4) is a formal consequence of Lemma 1.2.9 (2).

□

Chapter 2

Motives of commutative group schemes

This chapter is directly adapted, with slights changes in notation, from the upcoming paper [AHPL14], joint with G. Ancona and A.Huber.

Organization of this chapter and conventions

Section 2.1 deals with generalities on commutative group schemes. Section 2.2 introduces the motive $\Sigma^\infty G_\mathbb{Q}$ and gives its basic properties. Section 2.3 states the main theorem and its consequences. The proof is then given in Section 2.4. We study the Betti, ℓ-adic and Hodge realizations of the Künneth components in Section 2.5. In Section 2.6 we present the straightforward generalisation of our results to the setting of smooth commutative algebraic group spaces. The two appendices deal with technical points of the proof of the main theorem explained above. Appendix 2.A deals with the construction of a functorial cofibrant resolution for the sheaf $G \otimes \mathbb{Q}$. In Appendix 2.B we establish qfh-descent for the presheaf given by a commutative group.

Throughout let S be a noetherian scheme of finite dimension. Let Sm/S be the category of smooth S-schemes of finite type and Sch/S the category of S-schemes of finite type. We also write Sm and Sch when there is no ambiguity on the base. By sheaf we mean an étale sheaf on Sm/S, unless specified otherwise. By G, we denote either a group scheme or an algebraic group space that is always assumed to be commutative, smooth of finite type over S. In this situation, we often write G/S for concision. For any morphism $f : T \longrightarrow S$ we will write

$$G_T/T = G \times_S T/T$$

for the base change of G/S along f. We write $\mathbf{cGrp}_S$ for the category of smooth commutative group schemes of finite type over S.

2.1 Group schemes

Let S be a noetherian scheme of finite dimension and G/S a smooth commutative group scheme of finite type.

Since G/S is smooth, the union of the neutral components G_s^0 for $s \in S$ is open in G by [ABD$^+$65, Exposé VI$_B$ Théorème 3.10]. We write G^0 for the corresponding open subscheme of G and $\pi_0(G/S) := G/G^0$ for the quotient étale sheaf on Sch/S. We also write $\pi_0(G/S)$ for its restriction to Sm/S. By definition of G^0, the formation of G^0 and $\pi_0(G/S)$ for G/S smooth commutes with base change.

The following statement uses the notion of an algebraic space. For basics on algebraic spaces, see Section 2.6 below.

Lemma 2.1.1. *The scheme G^0 is a smooth S-group scheme of finite type and $\pi_0(G/S)$ is an étale algebraic group space. If G^0 is closed in G (for instance if S is the spectrum of a field) then*

$\pi_0(G/S)$ is a scheme. Moreover, there is a canonical short exact sequence of étale sheaves

$$0 \longrightarrow G^0 \longrightarrow G \xrightarrow{p_G} \pi_0(G/S) \longrightarrow 0 \ .$$

Proof. Since G/S is smooth of finite type and S is noetherian, G^0 is also smooth and of finite type.

As the quotient of the smooth equivalence relation $G^0 \times_G (G \times_S G)$, the sheaf G/G^0 is an S-algebraic group space (see e.g [LMB00, Corollaire 10.4]).

The algebraic group space $\pi_0(G/S)$ is locally of finite presentation (because both G and G^0 are locally of finite presentation) and formally étale (let $T^0 \to T$ be an infinitesimal thickening; then $(G/G^0)(T) \to (G/G^0)(T^0)$ is surjective because G is smooth, and injective because G^0 is open in G). We conclude that $\pi_0(G/S)$ is an étale algebraic space.

If G^0 is open and closed in G then $\pi_0(G/S)$ is separated. A quasi-finite separated algebraic space of finitve presentation is a scheme by [Knu71, II, Theorem 6.15]. ☐

The function $s \mapsto |\pi_0(G_{\bar{s}}/\bar{s})|$ (where $\bar{s}$ is any geometric point above s) is locally constructible on S by [Gro66b, Corollaire 9.7.9]. In particular, it is bounded since S is quasi-compact. This justifies the following definition.

Définition 2.1.2. The *order* of $\pi_0(G/S)$, denoted $o(\pi_0(G/S))$, is defined as the least common multiple of the order of all the elements of the groups $\pi_0(G_{\bar{s}}/\bar{s})$ (with $\bar{s}$ geometric point of S).

Définition 2.1.3. For any point $s \in S$, we write G_s for the fibre, g_s for its abelian rank and r_s for its torus rank. The *Kimura dimension* of G/S is:

$$\mathrm{kd}(G/S) := \max\{2g_s + r_s | s \in S\}$$

This terminology will be justified by Theorem 2.3.3.

Lemma 2.1.4. *The value* $\mathrm{kd}(G/S)$ *is the maximum of* $2g_s + r_s$ *for* s *varying in generic points of* S.

Proof. After replacing G by G^0, we can suppose that G is fibrewise connected. Let us fix $t \in S$. After base changing to the strict henselization of the local ring at t, we can assume S is strict henselian. It is enough to show that, under this hypothesis, $2g_s + r_s \geq 2g_t + r_t$ for all $s \in S$.

Let us fix a prime ℓ which is coprime to the residual characteristic of S and, for all natural n, consider the multiplication map $[\ell^n] : G \to G$. The integer $2g_s + r_s$ is the rank of the ℓ-adic Tate module of G_s. By [BLR90a, §7.3 Lemma 2(b)], the group scheme $\ker[\ell^n]$ is étale over S. So, by Hensel's Lemma, any section of this étale morphism at t extends to a section over S. In particular the rank of the Tate module of G_s has its minimum at $s = t$. ☐

2.2 The 1-motive $\Sigma^\infty G_{\mathbb{Q}}$

Définition 2.2.1. We write G for the étale sheaf of abelian groups on Sm/S defined by G, i.e.,

$$G(Y) = \mathrm{Mor}_{\mathsf{Sm}/S}(Y, G)$$

for $Y \in \mathsf{Sm}/S$.

Let

$$a_{G/S} : \mathbb{Z}\,\mathrm{Mor}_S(\cdot, G) \to G$$

be the morphism of presheaves of abelian groups on Sm/S induced by the addition map

We may omit S from these notations if the base scheme is clear.

Remark 2.2.2. By étale descent, G is a sheaf; $G_{\mathbb{Q}}$ is then also a sheaf by a quasi-compactness argument [AEWH13, Lemma 2.1.2]. However, the presheaves $\mathbb{Z}\,\mathrm{Mor}_S(\cdot, G)$ and $\mathbb{Q}\,\mathrm{Mor}_S(\cdot, G)$ are not sheaves.

Définition 2.2.3. 1. We write

$$G_{\mathbb{Q}} \in \mathbf{DA}^{\mathrm{eff}}(S)$$

to be the motive induced by the sheaf $G_{\mathbb{Q}}$ and

$$\Sigma^\infty G_{\mathbb{Q}} = \Sigma^\infty G_{\mathbb{Q}} \in \mathbf{DA}(S) \ .$$

2. Define
$$\alpha^{\mathrm{eff}}_{G/S} : M^{\mathrm{eff}}_S(G) \longrightarrow G_{\mathbb{Q}}$$
to be the morphism in $\mathbf{DA}(S)$ induced by the sheafification of $a_{G/S} \otimes \mathbb{Q}$ and
$$\alpha_{G/S} = \Sigma^\infty \alpha^{\mathrm{eff}}_{G/S} : M_S(G) \longrightarrow \Sigma^\infty G_{\mathbb{Q}}$$

Remark 2.2.4. The reader should compare with [AEWH13, Definition 2.1.5] which is an effective analogue with transfers over a perfect field. A comparison between the two morphisms will be made in Section 2.4.1. The definition in *op. cit.* cannot be generalized over a general S, because it is not yet known that finite correspondences from X to Y are related to morphisms from X to a symmetric power of Y. This point is studied in the upcoming work of Harrer [Har].

Remark 2.2.5. The assignment
$$G/S \mapsto \Sigma^\infty G_{\mathbb{Q}}$$
defines a functor
$$\mathbf{cGrp}_S \longrightarrow \mathbf{DA}(S) \ .$$

Lemma 2.2.6. *1. Short exact sequences in $\mathbf{cGrp}_S$ are sent to exact triangles in $\mathbf{DA}(S)$.*

2. The inclusion $G^0 \to G$ induces an isomorphism $\Sigma^\infty G_{\mathbb{Q}} \simeq G^0_{\mathbb{Q}}$.

3. The morphism $\alpha_{G/S}$ is natural in $G \in \mathbf{cGrp}_S$.

Proof. 1. The functor from $\mathbf{cGrp}_S$ to the category of étale sheaves of $\mathbb{Q}$-vector spaces on Sm/S which sends G to $G_{\mathbb{Q}}$ is $\mathbb{Z}$-linear and exact. The statement then follows from the construction of $\mathbf{DA}(S)$.

2. As the quotient G/G^0 is a torsion sheaf the map $G^0 \to G$ induces an isomorphism of sheaves $G^0_{\mathbb{Q}} \xrightarrow{\sim} G_{\mathbb{Q}}$.

3. The morphism $a_{G/S} \otimes \mathbb{Q}$ is natural in $G \in \mathbf{cGrp}_S$, and so is its sheafification.

□

Proposition 2.2.7. *Let $f : T \to S$ be a morphism and $G \to S$ a smooth commutative group scheme. Then with the notation of Definition 2.2.3 there are canonical isomorphisms*
$$\underline{f}^* G_{\mathbb{Q}} \simeq G_{T\mathbb{Q}}$$

and, modulo these isomorphisms, we have equalities of morphisms
$$\underline{f}^*(\alpha^{(\mathrm{eff})}_{G/S}) = \alpha^{(\mathrm{eff})}_{G_T/T} \ .$$

Proof. Since pullbacks commute with suspension (Lemma 1.2.5(3)) it is enough to treat the effective case. In this proof, we write for clarity $\underline{f}^*$ for the underived pullback on complexes of sheaves and $f^* : \mathbf{DA}^{\mathrm{eff}}(S) \to \mathbf{DA}^{\mathrm{eff}}(T)$ for the triangulated pullback functor.

By the universal property of fibre products, we have
$$\underline{f}^* G = G \times_S T \ .$$

By applying Theorem 2.A.1 to the sheaf G and switching to cohomological indexing we obtain a complex $A(G) \in \mathbf{Cpl}_{\geq 0}(\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}/S, \mathbb{Z})$ together with a map
$$r : A(G) \to G$$

with the following properties.

1. For all $i \in \mathbb{N}$, the sheaf $A(G)_i$ is of the form $\bigoplus_{j=0}^{d(i)} \mathbb{Z}(G^{a(i,j)})$ for some $d(i), a(i,j) \in \mathbb{N}$. In particular $A_{\mathbb{Q}}(G) := A(G) \otimes \mathbb{Q}$ is a bounded above chain complex of sums of representable sheaves of $\mathbb{Q}$-vector spaces; hence it is cofibrant in the projective model structure that we use to define $\mathbf{DA}^{\mathrm{eff}}(S)$.

2. There is a morphism $\tilde{a}_{G/S} : \mathbb{Z}(G) \to A(G)$ lifting $a_{G/S}$.

3. The map $r_{\mathbb{Q}} := r \otimes \mathbb{Q}$ is a quasi-isomorphism.

There is a complex $A(G_T) \in \mathbf{Cpl}_{\geq 0}(\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}/T, \mathbb{Z}))$ and a map $r_T : A(G_T) \to G_T$ with the same properties, and all those objects are compatible with pullbacks. Putting all of this together, we obtain canonical isomorphisms of objects

$$f^*\Sigma^\infty G_{\mathbb{Q}} \xleftarrow[\sim]{f^*(r_{\mathbb{Q}})} \underline{f}^*(A_{\mathbb{Q}}(G)) \simeq A_{\mathbb{Q}}(G_T) \xrightarrow[\sim]{r_{\mathbb{Q}}} \Sigma^\infty G_{T\mathbb{Q}}$$

and, modulo these isomorphisms, equalities of morphisms:

$$f^*\alpha^{\text{eff}}_{G/S} = \underline{f}^*(\tilde{a}_{G/S} \otimes \mathbb{Q}) = \tilde{a}_{G_T/T} \otimes \mathbb{Q} = \alpha^{\text{eff}}_{G_T/T}$$

This finishes the proof. □

The rest of the section is devoted to study the 1-motive *with transfers* of G and to compare it to $\Sigma^\infty G_{\mathbb{Q}}$. It will be used only in section 2.4 and can be skipped on a first reading.

Theorem 2.2.8. *Let G be a smooth commutative group scheme over an excellent (noetherian and finite dimensional) scheme S. The étale sheaf $G_{\mathbb{Q}}$ on Sm/S represented by G has a unique structure of étale sheaf with transfers, which we denote $G^{\text{tr}}_{\mathbb{Q}}$. Moreover, there is a unique map*

$$a^{\text{tr}}_{G/S} : \mathbb{Q}(G)^{\text{tr}} \to G^{\text{tr}}_{\mathbb{Q}}$$

of sheaves of transfers extending $a_{G/S}$.

Proof. This comes from the fact that $G_{\mathbb{Q}}$ is a qfh-sheaf (Proposition 2.B.2) and that such sheaves have a unique structure of sheaves with transfers (as follows from Theorem 1.2.10 and the Yoneda lemma). The morphism $a^{\text{tr}}_{G/S}$ is defined by taking the qfh-sheafification of $a_{G/S}$ and using the natural isomorphisms of Theorem 1.2.10 and Proposition 2.B.2. Uniqueness follows again from Yoneda. □

We can then proceed as in the case of $\Sigma^\infty G_{\mathbb{Q}}$.

Définition 2.2.9. 1. Define the (effective) 1-motive of G

$$G^{\text{tr}}_{\mathbb{Q}} \in \mathbf{DM}^{(\text{eff})}(S)$$

to be the motive induced by the sheaf with transfers $G^{\text{tr}}_{\mathbb{Q}}$ (analogously to Definition

2. Define
$$\alpha^{\text{tr}}_{G/S} : M_S(G)^{\text{tr}} \longrightarrow \Sigma^\infty G^{\text{tr}}_{\mathbb{Q}}$$
to be the morphism in $\mathbf{DM}(S)$ induced by $a^{\text{tr}}_{G/S}$.

Recall the adjoint pairs of functors (see §1.2.2)

$$\underline{a}_{\text{tr}} : \mathbf{Sh}_{\text{ét}}(\mathsf{Sm}) \rightleftarrows \mathbf{Sh}_{\text{ét}}(\mathsf{Sm}^{\text{tr}}) : \underline{o}^{\text{tr}}$$

and

$$a_{\text{tr}} : \mathbf{DA}^{\text{eff}}(S) \rightleftarrows \mathbf{DM}^{\text{eff}}(S) : o_{\text{tr}} \ .$$

Proposition 2.2.10. *Let S be excellent. Let G be a smooth commutative group scheme over S.*

1. *The natural morphism induced from the counit of $\underline{a}_{\text{tr}} \dashv \underline{o}^{\text{tr}}$ for étale sheaves*

$$\eta_G : \underline{a}_{\text{tr}}\underline{o}^{\text{tr}} G^{\text{tr}}_{\mathbb{Q}} \to G^{\text{tr}}_{\mathbb{Q}}$$

is an isomorphism of sheaves with transfers.

2. The natural morphism (induced from the counit of $a_{\mathrm{tr}} \dashv o_{\mathrm{tr}}$ for effective motives, followed by infinite suspension in the stable case)

$$L\eta_G^{(\mathrm{eff})} : a_{\mathrm{tr}} G_{\mathbb{Q}} \to G_{\mathbb{Q}}^{\mathrm{tr}}$$

is an isomorphism.

Proof. We have

$$a_{\mathrm{tr}}\Sigma^{\infty} G_{\mathbb{Q}} = a_{\mathrm{tr}}\Sigma^{\infty} G_{\mathbb{Q}} \simeq \Sigma^{\infty}_{\mathrm{tr}} a_{\mathrm{tr}} G_{\mathbb{Q}}$$

by Lemma 1.2.16 (2). Since this isomorphism is compatible with the counit maps $L\eta_G$ and $L\eta_G^{\mathrm{eff}}$, it is enough to treat the effective case in (2). We first reduce to item (1) as follows.

We apply Theorem 2.A.1 to the sheaf on Sch/S represented by G (that we also denote by G for simplicity). This yields a resolution

$$r_{\mathbb{Q}} : A_{\mathbb{Q}}(G) \longrightarrow \underline{o}^{\mathrm{tr}}(G_{\mathbb{Q}})^{\mathrm{tr}}$$

where $A_{\mathbb{Q}}(G)$ is a bounded above complex of sheaves of $\mathbb{Q}$-vector spaces whose terms are finite sums of sheaves represented by smooth S-schemes. Hence $(r_{\mathbb{Q}})_{|\mathsf{Sm}}$ is a cofibrant resolution of $G_{\mathbb{Q}}$ in the projective model structure on complexes of sheaves on Sm/S. By definition of the derived counit of a Quillen adjunction, the counit map $L\eta_G^{\mathrm{eff}}$ is the image of the following composition at the model category level:

$$\underline{a}_{\mathrm{tr}} A_{\mathbb{Q}}(G)_{|\mathsf{Sm}} \xrightarrow{\underline{a}_{\mathrm{tr}}(r_Q)_{\mathsf{Sm}}} \underline{a}_{\mathrm{tr}}\underline{o}^{\mathrm{tr}}(G_{\mathbb{Q}})^{\mathrm{tr}} \xrightarrow{\eta_G} (G_{\mathbb{Q}})^{\mathrm{tr}} .$$

The assumptions of Lemma 1.2.11 are satisfied for $A_{\mathbb{Q}} \xrightarrow{r_{\mathbb{Q}}} G_{\mathbb{Q}}$. By point ((i)) of that lemma, the first map of the above composition is a quasi-isomorphism. It thus remains to show that the second map is an isomorphism, which is precisely item (1).

The functor $\underline{o}^{\mathrm{tr}}$ at the level of sheaves with transfers is conservative so it is enough to show that $\underline{o}^{\mathrm{tr}}\eta_G$ is an isomorphism. By Lemma 1.2.11 ((ii)) and the triangular identity of adjunctions we have the following commutative diagram:

$$\begin{tikzcd}
\underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}G_{\mathbb{Q}} \arrow[r, "\underline{o}^{\mathrm{tr}}\eta_G"'] \arrow[rr, bend left, "\mathrm{id}"] & G_{\mathbb{Q}} \arrow[r, "\epsilon"'] \arrow[dr, "\epsilon'"'] & \underline{o}^{\mathrm{tr}}\underline{a}_{\mathrm{tr}}G_{\mathbb{Q}} \\
 & & (G_{\mathbb{Q}})_{\mathrm{qfh}|\mathsf{Sm}}. \arrow[u, "\sim"']
\end{tikzcd}$$

Proposition 2.B.2 shows that the map ϵ' is an isomorphism. Together with the diagram, this finishes the proof. ☐

2.3 Statement of the main theorem

Définition 2.3.1. Let $G_{\mathbb{Q}} \in \mathbf{DA}^{\mathrm{eff}}(S)$, $\alpha_{G/S}^{(\mathrm{eff})}$ be as in Definition 2.2.3. For any integer $n \geq 0$ write Δ_G^n for the n-fold diagonal immersion.

1. We define $\varphi_{n,G}^{(\mathrm{eff})}$ to be the morphism

$$\varphi_{n,G}^{(\mathrm{eff})} : M_S^{(\mathrm{eff})}(G) \xrightarrow{M_S^{(\mathrm{eff})}(\Delta_G^n)} M_S^{(\mathrm{eff})}(G)^{\otimes n} \xrightarrow{\alpha_{G/S}^{(\mathrm{eff})\otimes n}} G_{\mathbb{Q}}^{\otimes n}.$$

   As Δ_G^n is invariant under permutations, this factors uniquely

$$\begin{tikzcd}
M^{(\mathrm{eff})}(G) \arrow[rr, "\alpha_{G/S}^{(\mathrm{eff})\otimes n} \circ M_S^{(\mathrm{eff})}(\Delta_G^n)"] \arrow[dr, "\varphi_{n,G}^{(\mathrm{eff})}"'] & & G_{\mathbb{Q}}^{\otimes n} \\
 & \mathsf{Sym}^n(G_{\mathbb{Q}}) \arrow[ur] &
\end{tikzcd}$$

2. Let $\mathrm{kd}(G/S)$ be the integer in Definition 2.1.3. Define the morphism

$$\varphi_{G/S}^{(\mathrm{eff})} = \sum_{n=0}^{\mathrm{kd}(G/S)} \varphi_{n,G}^{(\mathrm{eff})} : M_S^{(\mathrm{eff})}(G) \longrightarrow \bigoplus_{n=0}^{\mathrm{kd}(G/S)} \mathsf{Sym}^n(G_{\mathbb{Q}}).$$

3. Let p_G be as in Lemma 2.1.1. Define the morphism

$$\psi_{G/S}^{(\mathrm{eff})} : M_S^{(\mathrm{eff})}(G) \longrightarrow \left(\bigoplus_{n=0}^{\mathrm{kd}(G/S)} \mathsf{Sym}^n G_{\mathbb{Q}} \right) \otimes M_S^{(\mathrm{eff})}(\pi_0(G/S))$$

   to be $\psi_{G/S}^{(\mathrm{eff})} = (\varphi_{G/S}^{(\mathrm{eff})} \otimes M_S^{(\mathrm{eff})}(p_G)) \circ M_S^{(\mathrm{eff})}(\Delta_G^2)$.

Remark 2.3.2. The Künneth formula holds by Proposition 1.2.5(1). This is used in the definition above to get a morphism

$$M_S^{(\mathrm{eff})}(\Delta_G^n) : M_S^{(\mathrm{eff})}(G) \to M_S^{(\mathrm{eff})}(G \times_S \cdots \times_S G) = M_S^{(\mathrm{eff})}(G)^{\otimes n} \ ,$$

as well as to define $\psi_{G/S}^{(\mathrm{eff})}$.

The morphisms $\varphi_G^{(\mathrm{eff})}$ and $\psi_{G/S}^{(\mathrm{eff})}$ are the unique extensions of α_G compatible with the natural comultiplication on both sides; see [AEWH13, Section 3.2] for a more detailed discussion.

The main result of the paper is the following theorem.

Theorem 2.3.3. *Let S be a noetherian finite dimensional scheme and G/S a smooth commutative group scheme of finite type over S.*

For $m \in \mathbb{Z}$, let $[m] : G \to G$ be the morphism of multiplication by m. Let $\Sigma^\infty G_{\mathbb{Q}} \in \mathbf{DA}(S)$ be the motive from Definition 2.2.3, $\psi_{G/S}$ be the map as in Definition 2.3.1, $o(\pi_0(G/S))$ and $\mathrm{kd}(G/S)$ be the integers in Definitions 2.1.2 and 2.1.3. Then the following statements hold.

1. *The relative motive $\Sigma^\infty G_{\mathbb{Q}}$ is odd of Kimura dimension $\mathrm{kd}(G/S)$ (i.e. $\mathsf{Sym}^n \Sigma^\infty G_{\mathbb{Q}}$ is zero for $n > \mathrm{kd}(G/S)$ and non zero otherwise).*

2. *The map*

$$\psi_{G/S} : M_S(G) \longrightarrow \left(\bigoplus_{n=0}^{\mathrm{kd}(G/S)} \mathsf{Sym}^n \Sigma^\infty G_{\mathbb{Q}} \right) \otimes M(\pi_0(G/S))$$

   is an isomorphism of motives. It is natural in $G \in \mathbf{cGrp}_S$, it commutes with base change (in S) and it respects the natural structures of Hopf algebras.

3. *The motives $\Sigma^\infty G_{\mathbb{Q}}$ and $M(\pi_0(G/S))$ are geometric motives and the motive $M_S(G)$ is finite dimensional in the sens of [Kim05].*

4. *The direct factor*

$$\mathfrak{h}_n(G/S) = \psi_{G/S}^{-1} \left(\mathsf{Sym}^n \Sigma^\infty G_{\mathbb{Q}} \otimes M_S(\pi_0(G/S))\right)$$

   of $M_S(G)$ is intrinsically characterized as follows: for $m \in \mathbb{Z}$ that is equal to 1 modulo $o(\pi_0(G/S))$, the map $M_S([m])$ operates on $\mathfrak{h}_n(G/S)$ as $m^n \mathrm{id}$.

Remark 2.3.4. $M_S(G)$ and $M(\pi_0(G/S))$ carry a Hopf algebra structure because G/S and $\pi_0(G/S)$ are group objects. The Hopf algebra structure on $\bigoplus_{n \geq 0} \mathsf{Sym}^n \Sigma^\infty G_{\mathbb{Q}}$ is the one of the *symmetric coalgebra*; see [AEWH13, Appendix B]. It is isomorphic but not identical to the symmetric Hopf algebra.

Remark 2.3.5. We expect the morphism $\psi_{G/S}^{\mathrm{eff}}$ to be already an isomorphism. Most steps of the proof take place in $\mathbf{DA}^{\mathrm{eff}}(S)$. When S is of characteristic 0, the result of [Ayoarb, Annexe B.] can be used to show that $\psi_{G/S}^{\mathrm{eff}}$ is an isomorphism; since this requires some extensions of results from stable to effective motives, which are easy but not in the literature, we do not write the proof here. An effective proof in general would require a comparison between effective étale motives with and without transfers over an arbitrary field. An alternative approach would be to try to redo [AEWH13] in the category of motives without transfers. There the missing ingredient is a special case of the comparison, namely a transfer-free computation of the effective motivic cohomology of curves.

Remark 2.3.6. The theorem above holds in the more general context of commutative algebraic group spaces; see Section 2.6.

As a consequence, we also get a version of the theorem for motives with transfers.

Theorem 2.3.7. *Let S be an excellent (noetherian and finite dimensional) scheme and G/S a smooth commutative group scheme of finite type over S. Let $\psi_{G/S}$ be the map as in Definition 2.3.1 and a_{tr} the functor defined in §1.2.2. Let $M_S(G)^{\mathrm{tr}}$ and $M_S(\pi_0(G/S))^{\mathrm{tr}}$ be the relative motives with transfers of G and $\pi_0(G/S)$ and $\Sigma^\infty G_{\mathbb{Q}}^{\mathrm{tr}} \in \mathbf{DM}(S)$ as in Definition 2.2.9. Then*

$$\psi_G^{\mathrm{tr}} : M_S(G)^{\mathrm{tr}} \xrightarrow{\sim} \left(\bigoplus_{n=0}^{\mathrm{kd}(G/S)} \mathsf{Sym}^n \Sigma^\infty G_{\mathbb{Q}}^{\mathrm{tr}} \right) \otimes M_S(\pi_0(G/S))^{\mathrm{tr}}$$

is an isomorphism and the analogous of Theorem 2.3.3 holds.

Proof. First, apply the functor a_{tr} to the isomorphism in Theorem 2.3.3. Its image is of course an isomorphism. One concludes by the following proposition. □

Proposition 2.3.8. *Let S be excellent. Modulo the isomorphisms of Proposition 2.2.10, $\underline{a}_{\mathrm{tr}}$ sends the morphism $a_{G/S}$ to $a_{G/S}^{\mathrm{tr}}$ and a_{tr} sends the morphism α_G to α_G^{tr}, the morphism ϕ_G to ϕ_G^{tr} and the morphism ψ_G to ψ_G^{tr}.*

Proof. The construction of ϕ_G and ψ_G from α_G together with the fact that a_{tr} is monoidal (Lemma 1.2.16 (4)) shows that it is enough the statement for α_G. As in the proof of Proposition 2.2.10, it is then enough to treat the effective statement.

We have a natural commutative diagram of sheaves without transfers

$$\begin{tikzcd} \mathbb{Q}(G) \arrow[r, "\alpha_G"] \arrow[d] & G_{\mathbb{Q}} \arrow[d, "\sim"] \\ \underline{o}^{\mathrm{tr}}\mathbb{Q}(G)^{\mathrm{tr}} \arrow[r, "\underline{o}^{\mathrm{tr}}\alpha_G^{\mathrm{tr}}"] & \underline{o}^{\mathrm{tr}}(G_{\mathbb{Q}})^{\mathrm{tr}} \end{tikzcd}$$

where the left vertical map is the unit map $\epsilon_{\mathbb{Q}(G)}$. This diagram induces a commutative diagram in $\mathbf{DA}^{\mathrm{eff}}(S)$ whose left vertical map is the derived unit map $\epsilon_{M_S^{\mathrm{eff}}(G)}$. We apply the functor a_{tr}, and stack the resulting diagram with a diagram coming from the naturality of the derived counit maps:

$$\begin{tikzcd} M_S^{\mathrm{eff}}(G)^{\mathrm{tr}} \arrow[r, "a_{\mathrm{tr}}\alpha_G"] \arrow[d, "a_{\mathrm{tr}}\epsilon_{M_S^{\mathrm{eff}}(G)}"'] & a_{\mathrm{tr}}G_{\mathbb{Q}} \arrow[d, "\sim"] \\ a_{\mathrm{tr}}\underline{o}^{\mathrm{tr}}M_S^{\mathrm{eff}}(G)^{\mathrm{tr}} \arrow[r, "a_{\mathrm{tr}}\underline{o}^{\mathrm{tr}}\alpha_G^{\mathrm{tr}}"] \arrow[d, "\eta_{M_S^{\mathrm{eff}}(G)^{\mathrm{tr}}}"'] & a_{\mathrm{tr}}o^{\mathrm{tr}}M_1^{\mathrm{eff}}(G)^{\mathrm{tr}} \arrow[d, "\eta_{G_{\mathbb{Q}}^{\mathrm{tr}}}"', "\sim"] \\ M_S^{\mathrm{eff}}(G)^{\mathrm{tr}} \arrow[r, "\alpha_G^{\mathrm{tr}}"] & M_1^{\mathrm{eff}}(G)^{\mathrm{tr}}. \end{tikzcd}$$

The left vertical composition is the identity by the triangular identity for the object $M_S^{\mathrm{eff}}(G)$ and the right vertical composition is the isomorphism of Proposition 2.2.10. This finishes the proof. □

Remark 2.3.9.
1. Using Theorem 1.2.10 one can give an alternative description of $a_{\mathrm{tr}}\psi_{G/S}$ using qfh sheafification. Recall that the morphism $\psi_{G/S}$ is formally constructed from a morphism of sheaves $a_{G/S} \otimes \mathbb{Q}$ (Definition 2.2.1). If one replaces in this formal construction $a_{G/S} \otimes \mathbb{Q}$ by its qfh sheafification and $\mathbf{DA}(S)$ by $\mathbf{DM}(S)$ one ends with $a_{\mathrm{tr}}\psi_{G/S}$.

2. By the work of Cisinski and Déglise, the different categories of motives are related by functors. By replacing a_{tr} by those functors one obtains analogous results in the other categories of motives.

   More precisely, for the category $\mathbf{DM}_h(S, \mathbb{Q})$ of h-motives use [CD09, Theorem 16.1.2] (one needs to suppose S excellent noetherian and finite dimensional) and for the category $\mathbf{DM}_{\mathrm{B}}(S)$ of Beilinson motives use [CD09, Theorem 16.2.18] (S noetherian and finite dimensional).

Theorem 2.3.10. *Suppose that S is regular. The i-th Chow group of G with rational coefficients decomposes as*

$$\mathcal{H}^i(G)_{\mathbb{Q}} = \bigoplus_{j=0}^{\mathrm{kd}(G/S)} \mathcal{H}^i_j(G),$$

where

$$\mathcal{H}^i_j(G) = \{Z \in \mathcal{H}^i(G)_{\mathbb{Q}} \,|\, [m]^* Z = m^j Z \,, \forall m \equiv 1 \,(\mathrm{mod}\ o(\pi_0(G/S)))\} \ .$$

Proof. Since S is regular noetherian, by [CD09, Corollary 14.2.14] (and [Ful98, 20.1] for the comparison between Chow groups and K-theory), there is a canonical isomorphism

$$\mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}(S)}(M_S(G), \mathbb{Q}(i)[2i]) \cong \mathcal{H}^i(G)_{\mathbb{Q}} \ .$$

Moreover, there is a canonical equivalence between $\mathbf{DM}_{\mathrm{B}}(S)$ and $\mathbf{DA}(S)$ [CD09, Theorem 16.2.18]. Hence the decomposition $M_S(G) = \bigoplus_{i=0}^{\mathrm{kd}(G/S)} \mathfrak{h}_n(G/S)$ of Theorem 2.3.3 implies the one in the statement. ☐

Remark 2.3.11.
1. In the same way one has a decomposition of the higher Chow groups or of the Suslin homology.
2. Some of the eigenspaces $\mathcal{H}^i_j(G)$ above should be zero. For example, if S is the spectrum of a field and G is an abelian variety of dimension g, then Beauville [Bea86] proved that $\mathcal{H}^i_j(G)$ is zero if $j < i$ or if $j > g + i$, and that, in general, $\mathcal{H}^i_j(G)$ is non-zero for $i \leq j \leq 2i$. Conjecturally $\mathcal{H}^i_j(G)$ should be zero if $j > 2i$, this is part of the (still open) Bloch-Beilinson-Murre conjecture.

   In our general setting one can hope to show similar vanishings of some of the eigenspaces $\mathcal{H}^i_j(G)$ following the methods of Sugiyama [Sug13] for semiabelian varieties.

2.4 Proof of the main theorem

2.4.1 The case over a perfect field

In this section let $S = \mathbf{Spec}(k)$ be the spectrum of a perfect field. Recall that in this case $\pi_0(G/k)$ is a group scheme by Lemma 2.1.1 so we do not have to consider motives of algebraic spaces in this section. We show Theorem 2.3.3 for all commutative group schemes of finite type G over k. This is essentially the main result of [AEWH13], with exception that in *loc. cit.* the authors work in the effective category of motives with transfers and in this paper we primarily work in the stable category of motives without transfers. The point is then to compare the two approaches.

Proposition 2.4.1. *Let G be a smooth commutative group scheme over k. Then $\Sigma^\infty G_{\mathbb{Q}}$ is odd of Kimura dimension* $\mathrm{kd}(G/k)$ *and the morphism $\psi_{G/k}$ is an isomorphism.*

Proof. By Theorem 1.2.13 and Proposition 2.2.10 it suffices to show that

$$a_{\mathrm{tr}}\mathsf{Sym}^n(\Sigma^\infty G_{\mathbb{Q}}) = \mathsf{Sym}^*(a_{\mathrm{tr}}\Sigma^\infty G_{\mathbb{Q}}) = \mathsf{Sym}^n(\Sigma^\infty G^{\mathrm{tr}}_{\mathbb{Q}})$$

vanishes for $n > \mathrm{kd}(G)$ and that

$$a_{\mathrm{tr}}\psi_{G/k} = \psi^{\mathrm{tr}}_{G/k}$$

is an isomorphism.

As $\Sigma^\infty_{\mathrm{tr}}$ is monoidal (see 1.2.16(1)) and commutes with direct sums, we have an isomorphism

$$\bigoplus_{n=0}^{k} \mathsf{Sym}^n G^{\mathrm{tr}}_{\mathbb{Q}} \simeq \Sigma^\infty_{\mathrm{tr}} \left(\bigoplus_{n=0}^{k} \mathsf{Sym}^n G^{\mathrm{tr}}_{\mathbb{Q}} \right).$$

Modulo this isomorphism, ψ^{tr}_G agrees with the morphism deduced by applying $L\Sigma^\infty_{\mathrm{tr}}$ to

$$\psi^{\mathrm{eff,tr}}_{G/k} : M^{\mathrm{eff}}_k(G)^{\mathrm{tr}} \longrightarrow \left(\bigoplus_{n\geq 0} \mathsf{Sym}^n G^{\mathrm{tr}}_{\mathbb{Q}} \right) \otimes M^{\mathrm{eff}}_k(\pi_0(G/k))^{\mathrm{tr}} .$$

On the other hand, by the uniqueness part in Theorem 2.2.8, the morphism $\psi^{\mathrm{eff,tr}}_{G/k}$ is exactly the morphism considered in [AEWH13, §7.4] (although with different notation). We are now able to apply [AEWH13, Theorem 7.4.6], which concludes the proof. □

2.4.2 The general case

We return to an arbitrary base scheme S. Here is the result which is at the heart of the reduction.

Lemma 2.4.2. *1. Let $M \in \mathbf{DA}(S)$ be a motive. Then M is zero if and only if the pullback $i^*_{\bar{s}}M$ to any geometric point*

$$i_{\bar{s}} : \bar{s} \longrightarrow S$$

is zero.

2. *Let f be a morphism in $\mathbf{DA}(S)$. Then f is an isomorphism if and only if the pullback $i^*_{\bar{s}}(f)$ to any geometric point $\bar{s} \to S$ is an isomorphism.*

Proof. The second statement follows from the first by the axioms of a triangulated category.

We turn to the proof of the first. By [Ayo14b, Proposition 3.24] the family i^*_s for all points $s \in S$ is conservative. Hence, we may now assume that $S = \mathbf{Spec}\ k$ is the spectrum of a field. Let k^i be the inseparable closure of k. It is well-known that pull-back induces an equivalence of categories between the category of étale sheaves on $\mathbf{Spec}\ k$ and the category of étale sheaves on $\mathbf{Spec}\ k^i$. It is much more difficult, but nonetheless true, that it induces an equivalence between the categories of motives (see [CD09, Proposition 2.1.9] and [CD09, Theorem 14.3.3]).

We may now assume that k is perfect. Let $\bar{k}$ be an algebraic closure of k. Let $P \in \mathbf{DA}(\mathbf{Spec}\ k)$ such that the pull-back i^* to $\mathbf{Spec}\ \bar{k}$ vanishes. As the category $\mathbf{DA}(S)$ is compactly generated by [Ayo14b, Proposition 3.19], it suffices to show that all morphisms $f : M \to P$ with M compact vanish. By assumption, the morphism $i^*(f)$ vanishes. By [Ayo14b, Proposition 3.20] this implies that the pull-back of f to some finite separable extension of k vanishes. By [Ayo14b, Lemme 3.4] such a restriction is conservative and so $f = 0$. □

We can now complete the proof.

Proof of Theorem 2.3.3. 1. We consider the Kimura dimension $\mathrm{kd}(G/S)$ of $\Sigma^\infty G_{\mathbb{Q}}$. We claim that

$$\mathsf{Sym}^n(\Sigma^\infty G_{\mathbb{Q}}) = 0$$

for $n > \mathrm{kd}(G/S)$. By Lemma 2.4.2, we can test this after pullback to all geometric points $i_{\bar{s}} : \bar{s} \to S$. The functor $i^*_{\bar{s}}$ commutes with tensor product and hence with Sym^n. By Proposition 2.2.7 we have $i^*_{\bar{s}}\Sigma^\infty G_{\mathbb{Q}} = \Sigma^\infty G_{\bar{s}\mathbb{Q}}$. By definition $\mathrm{kd}(G/S) \geq \mathrm{kd}(G_{\bar{s}})$. Hence the vanishing holds by Proposition 2.4.1.

2. By Lemma 2.2.6(1) the morphism $\alpha_{G/S}$ is natural in $G \in \mathbf{cGrp}_S$. This implies naturality for the morphism $\psi_{G/S}$. Naturality implies that the morphism $\psi_{G/S}$ is a morphism of Hopf algebras by the same argument as in the absolute case; see [AEWH13, Proposition 3.2.9, Theorem 7.4.6].

   By Proposition 2.2.7 the morphism $\alpha_{G/S}$ commutes with base change. As the pullback on motives is a monoidal functor (Lemma 1.2.5 (4)) the morphism $\varphi_{G/S}$ also commutes with base change. Since the formation of $\pi_0(G/S)$ commutes with base change, so does $\psi_{G/S}$.

We turn to the claim that $\psi_{G/S}$ is an isomorphism. By Lemma 2.4.2, it suffices to check the assertion after pullback via $i_{\bar{s}} : \bar{s} \to S$ for all geometric points $\bar{s}$ of S. On the other hand $i_{\bar{s}}^* \psi_{G/S} = \psi_{G_{\bar{s}}/\bar{s}}$, as the map $\psi_{G/S}$ commutes with base change. This is an isomorphism by Proposition 2.4.1.

3. If $\pi_0(G/S) = S$, then $\Sigma^\infty G_{\mathbb{Q}}$ is geometric because it is a direct summand of a geometric object by part 2. This also implies the general case because $\Sigma^\infty G_{\mathbb{Q}} = \Sigma^\infty G^0_{\mathbb{Q}}$. Finally, $M(\pi_0(G/S))$ is geometric because it is a direct factor of $M_S(G)$ by part 2.

   Finite dimensionality is a notion stable by tensor product, finite sums and direct factors [And05, Lemme 3.7(3)], hence $M_S(G)$ is finite dimensional.

4. By Lemma 2.2.6(1), the multiplication by m map $[m]$ acts by $m \cdot \mathrm{id}$ on $\Sigma^\infty G_{\mathbb{Q}}$ for all integers m, so that $\mathsf{Sym}^n([m]) = m^n \mathrm{id}$ for all $n \in \mathbb{N}$. To conclude, note that if m is equal to 1 modulo $o(\pi_0(G/S))$, the multiplication by m is the identity on the space $\pi_0(G/S)$.

□

2.5 Realizations

We want to study the image of our decomposition under realization functors. We use the existence of realizations functors compatible with the six functors formalism. We have decided against axiomatizing the statement but rather treat the explicit cases of Betti and ℓ-adic realization.

2.5.1 Betti realization

In this section, we assume all schemes are of finite type over $\mathbb{C}$. For such a scheme S we denote by S^{an} the associated complex analytic space, equipped with its natural topology. Let $D(S^{\mathrm{an}}, \mathbb{Q})$ be the derived category of sheaves of $\mathbb{Q}$-vector spaces on S^{an} and $D^b_c(S^{\mathrm{an}}, \mathbb{Q})$ be the subcategory of bounded complexes with constructible cohomology. By [Ayo10a] (see also [CD09, 17.1.7.6] for an elaboration in terms of ring spectra) there is a system of covariant functors

$$R_B : \mathbf{DA}(S) \to D(S^{\mathrm{an}}, \mathbb{Q})$$

compatible with the six functors formalism on both sides (note that some commutativity morphisms are shown to be isomorphisms only when applied to constructible motives). It maps the Tate object $\mathbb{Q}(j)$ to $\mathbb{Q}$. Moreover, as the relative Betti homology of a smooth S-scheme lies in $D^b_c(S^{\mathrm{an}}, \mathbb{Q})$, constructible motives are sent to $D^b_c(S^{\mathrm{an}}, \mathbb{Q})$.

Proposition 2.5.1. *Let $\pi : G \to S$ be a smooth commutative group scheme of relative dimension d with connected fibres. Then:*

1. $R_B(M_S(G)) = \pi^{\mathrm{an}}_!(\pi^{\mathrm{an}})^! \mathbb{Q}_S = \pi^{\mathrm{an}}_! \mathbb{Q}_G[2d]$
2. $R_B(\Sigma^\infty G_{\mathbb{Q}}[-1]) = R^{2d-1}\pi^{\mathrm{an}}_! \mathbb{Q}_G$ *with fibre in $s \in S^{\mathrm{an}}$ given by $H_1(G_s, \mathbb{Q})$. We set $\mathcal{H}_1(G/S, \mathbb{Q}) = R_B(\Sigma^\infty G_{\mathbb{Q}}[-1])$.*
3. $\pi^{\mathrm{an}}_! \mathbb{Q}_G(d)[2d] = \bigoplus_{i=0}^{\mathrm{kd}(G/S)} (\bigwedge^i \mathcal{H}_1(G/S))[i]$.

Proof. First notice that π is separated by [ABD+65, Exposé VI$_B$ Corollaire 5.5] so that the exceptional operations $\pi_!$ and $\pi^!$ are well-defined. We have

$$M_S(G) = \pi_\# \mathbb{Q}_G = \pi_! \pi^! \mathbb{Q}_S$$

because π is smooth. The first assertion then follows from the same statement in $\mathbf{DA}(S)$ and the compatibility theorem [Ayo10a, Theoreme 3.19].

By our main Theorem 2.3.3, $R_B \Sigma^\infty G_{\mathbb{Q}}$ is a direct factor of $R_B(M_S(G)) \simeq \pi_! \pi^! \mathbb{Q}_S$. Its fibre over $s \in S^{\mathrm{an}}$ is given by $R_B(\Sigma^\infty G_{s\mathbb{Q}})$). This was computed in [AEWH13, Proposition 7.2.2] via a Hopf algebra argument which also applies in our setting with the difference that in *loc. cit.* the realization functor was assumed to be *contravariant*, i.e., $M(G_s)$ was mapped to $\pi_* \pi^* \mathbb{Q}$. In that language, the realization of $\Sigma^\infty G_{s\mathbb{Q}}$) was concentrated in degree 1 and equal to $H^1(G_s, \mathbb{Q})$ as a direct factor of $H^*(G_s, \mathbb{Q})$. Hence in the present setting, the realization is concentrated in degree -1 and equal to $H_1(G_s, \mathbb{Q})$.

This shows that indeed $R_B(\Sigma^\infty G_\mathbb{Q}[-1]) \simeq R^{2d-1}\pi_!^{\mathrm{an}}\mathbb{Q}_G$.

The last statement follows by functoriality of R_B. □

Remark 2.5.2.

1. The statement in Proposition 2.5.1 (3) means that our canonical motivic decomposition is a decomposition into relative Künneth components with respect to the conjectural standard motivic t-structure.
2. If G has constant abelian rank and torus rank over a regular scheme S, then $\mathcal{H}_1(G/S, \mathbb{Q})$ is a local system and thus (up to a shift) a perverse sheaf. We do not know if this is true in general and expect it to be false; however see the following example.

Example 2.5.3. Let S be regular of dimension 1 and G/S be a degenerating family of elliptic curves with multiplicative or additive reduction. In both cases $R_B(\Sigma^\infty G_\mathbb{Q})$ is a perverse sheaf.

2.5.2 ℓ-adic realization

In this section, we fix a prime ℓ and assume that all schemes are $\mathbb{Z}[\frac{1}{\ell}]$-schemes. For such a scheme S, let $D_c(S, \mathbb{Q}_\ell)$ be subcategory of complexes with constructible cohomology in the derived category of $\mathbb{Q}_\ell$-sheaves S in the sense of Ekedahl [Eke90]. By [Ayo14b, Section 9] there are covariant functors

$$R_\ell : \mathbf{DA}_c(S, \mathbb{Q}) \to D_c(S, \mathbb{Q}_\ell)$$

compatible with the six functors formalism on both sides. It maps the Tate object $\mathbb{Q}(j)$ to $\mathbb{Q}_\ell(j)$.

For a smooth group scheme G/S with connected fibres, we can deduce, by the same argument as in Proposition 2.5.1, an analoguous decomposition for $R_\ell(M_S(G))$. The corresponding object $\mathcal{H}_1(G/S, \mathbb{Q}_\ell)$ has fibre over $s \in S$ given by the rational Tate module $V_\ell(G_s)$. The analogues of Remark 2.5.2 also hold in this setting.

Remark 2.5.4. The ℓ-adic sheaf $\mathcal{H}_1(G/S, \mathbb{Q}_\ell)$ is given by the system of constructible torsion sheaves $G[\ell^n]$. This also holds integrally. Again this can be seen either directly in ℓ-adic cohomology or via a computation of the realization of $\Sigma^\infty G_\mathbb{Q}$.

2.5.3 Hodge realization

Let S be separated of finite type over $\mathbb{C}$. We expect the existence of Hodge realization functors

$$R_H : \mathbf{DA}_c(S) \to D^b(\mathrm{MHM}(S))$$

(where $\mathrm{MHM}(S)$ is Saito's category of mixed Hodge modules over S) compatible with the six functors formalism and such that the forgetful functor to the underlying derived category of sheaves is equal to R_B. This would yield a refinement of Proposition 2.5.1. Note that $R_H(\Sigma^\infty G_\mathbb{Q})$ is a mixed Hodge module (up to shift) if and only if $R_B(\Sigma^\infty G_\mathbb{Q})$ is a perverse sheaf (up to shift).

Ivorra has constructed a functor R_H for $S/\mathbb{C}$ smooth quasi-projective [Ivo14], but the compatibility with the six functors (in particular the tensor functor!) is still open. Hence we get a partial result. There is also upcoming work of Drew [Dre13b, Dre13a] defining a Hodge realization R'_H with values in some triangulated category $D_{\mathrm{Hdg}}(S)$ compatible with the six functors formalism. It is known that $D_{\mathrm{Hdg}}(\mathbb{C}) \simeq D^b(\mathrm{MHS})$ (so that R'_H puts in family the Hodge structures on the (co)homology of fibres) but the comparison with mixed Hodge modules over a more general S is still not understood.

2.6 Algebraic group spaces

We extend our main result from commutative group schemes to commutative algebraic group spaces. First we discuss generalities and some examples.

Recall (see e.g. [LMB00, Chapter 1]) that an algebraic space Y is an étale sheaf of sets over the site Sch/S with the étale topology induced by a scheme X and an étale equivalence relation R. This means that Y is the cokernel of the diagram of étale sheaves of sets

$$R \rightrightarrows X \ .$$

It is *smooth* (resp. *étale*) if X is smooth (resp. étale) over S. An algebraic group space is defined to be a group object in the category of algebraic spaces (which does not mean that we can find a presentation as above with X a group and R a subgroup).

Algebraic group spaces are closer to schemes than general algebraic spaces.

Proposition 2.6.1. *[Art69, Lemma 4.2] If S is the spectrum of a field and G/S an algebraic group space of finite type over S, then G is a scheme.*

By a spreading-out argument, one deduces the following seemingly stronger result.

Corollary 2.6.2. *Let S be a noetherian scheme and G/S an algebraic group space of finite type. Then exists a stratification of S such that G restricted to any stratum is a scheme.*

Commutative algebraic group spaces that are not group schemes appear naturally in algebraic geometry. We have already seen the example of the étale algebraic space $\pi_0(G/S)$ in Lemma 2.1.1, which was an instance of a very general result of Artin of representability of quotients [LMB00, Corollaire 10.4]. In the same vein, we have the following useful fact.

Lemma 2.6.3. *[FK88, Proposition 4.6] Let S be a scheme. The constructible étale sheaves of sets (resp. of abelian groups) are exactly the étale algebraic spaces (resp. étale algebraic group spaces) of finite type over S.*

A fundamental source of examples are Picard functors of proper flat cohomologically flat morphisms [BLR90a, 8.3]; they are not of finite type, smooth or separated in general, but see [BLR90a, 8.4].

Finally, the Weil restriction along a finite flat morphism sends (smooth) algebraic group spaces to (smooth) algebraic group spaces (see [BLR90a, 7.6] and [Ols06a, Theorem 1.5]).

For the rest of this section, we adopt the following shorthand

Définition 2.6.4. A *group space* is a smooth commutative algebraic group space of finite type.

Lemma 2.6.5. *Let G be a group space. Then there is a short exact sequence of group spaces*

$$0 \to G^0 \to G \to \pi_0(G/S) \to 0$$

such that all fibres of G^0 are connected and $\pi_0(G/S)$ is an étale group space. The formation of this sequence commutes with base change.

Proof. The union G^0 of neutral connected components of fibres is an open subset of the set of points of the algebraic space G by applying [Rom11, Proposition 2.2.1] to the zero section of G. Hence G^0 corresponds to an open group subspace of G. One then defines $\pi_0(G/S)$ as the quotient étale sheaf G/G^0 and proceeds as in the proof of Lemma 2.1.1. (The paper [Rom11] describes an étale group space $\pi_0(G/S)$ via its functor of points (see [Rom11, Definition 2.1.1.(1)] and [Rom11, 2.5.2.(i)]). One can show that it coincides with the one in the Lemma above using [Rom11, 2.5.2.(ii)].) $\square$

We define the Kimura dimension $\mathrm{kd}(G)$ of G and the order of $\pi_0(G/S)$ as in Section 2.1. The Kimura dimension is well-defined: assume G is represented by a smooth scheme X with relations in R. Then the fibre dimension of G is bounded by the fibre dimension of X and hence $\mathrm{kd}(G_s)$ is bounded by $2\dim_S X$. The order is well-defined by Lemma 2.6.5 and the fact that the sheaf $\pi_0(G/S)$ is constructible by Lemma 2.6.3.

Remark 2.6.6. We do not know if Lemma 2.1.4 holds in this more general setting. If G/S is separated, the proof given there works because the group spaces $G[l^n]$ are then schemes by [Knu71, II, Theorem 6.15]. There are non-separated group spaces over discrete valuation rings, but in standard examples the non-separatedness comes from removing connected components in the special fibre, which does not affect $\mathrm{kd}(G/S)$.

Let $M_S^{\mathrm{eff}}(G)$ (resp. $M_S(G)$) be the effective motive (resp. motive) of G as introduced in the conventions.

Lemma 2.6.7. *$M_S(G)$ is geometric.*

Proof. By Corollary 2.6.2 we have a stratification such that for any stratum $i : T \to S$, the restriction G_T is a scheme. We have $i^*M_S(G) = M_T(G_T)$ by Proposition 2.2.7. The result then follows by induction on strata and localisation. □

In analogy with the scheme case, we write $G_{\mathbb{Q}}$ for the image of $G_{|\mathsf{Sm}/S} \otimes \mathbb{Q}$ in $\mathbf{DM}^{\mathrm{eff}}(S)$ and $\Sigma^\infty G_{\mathbb{Q}}$ to be the object $\Sigma^\infty G_{\mathbb{Q}}$ in $\mathbf{DM}(S)$. The definitions of $\alpha_{G/S}$, $\phi_{G/S}$ and $\psi_{G/S}$ in Definition 2.2.3 and Definition 2.3.1 work without changes.

Theorem 2.6.8. *Let G be a group space in the sense of Definition 2.6.4. Then all assertions of Theorem 2.3.3 hold true.*

Proof. Let $f : T \to S$ be a morphism of schemes. First, we want to show that $f^*\Sigma^\infty G_{\mathbb{Q}} = \Sigma^\infty G_{T\mathbb{Q}}$. We use again the resolution of $\Sigma^\infty G_{\mathbb{Q}}$ obtained by Theorem 2.A.1. This time it is not clear if its terms are cofibrant. However, by Proposition 1.2.7, the pullback of $M_S(Y)$ is given by $M_T(Y_T)$ when Y is an algebraic space. Hence, the pull-back of the resolution of $\Sigma^\infty G_{\mathbb{Q}}$ agrees with the resolution of $\Sigma^\infty G_{T\mathbb{Q}}$, which implies $f^*\Sigma^\infty G_{\mathbb{Q}} = \Sigma^\infty G_{T\mathbb{Q}}$.

Now all steps in the proof work in the same way as in the scheme case, i.e, by reducing to the case of $S = \mathbf{Spec}\ k$ with k an algebraically closed field. By Proposition 2.6.1, we are then back to the case of group schemes settled by Proposition 2.4.1 (in particular we do not need to consider transfers on algebraic group spaces). □

Remark 2.6.9. Applying the realization functors R_B or R_ℓ, we also obtain a computation of the Künneth components. In particular, $R_B\Sigma^\infty G_{\mathbb{Q}}$ and $R_\ell\Sigma^\infty G_{\mathbb{Q}}$ are concentrated in degree -1. Conceptually this means that $\Sigma^\infty G_{\mathbb{Q}}$ should still be a homological 1-motive when G is a group space. Note, however, that the results are a bit weaker than in Section 2.5 because there are not as yet categories of triangulated motives over algebraic spaces satisfying the six functors formalism.

Appendix

2.A Eilenberg-MacLane complexes for sheaves

The main object of study in this paper is the sheaf G on Sm/S associated to a commutative algebraic group space G/S. In the course of the proof of the main theorem, we need to compute several left derived functors applied to the object $M_1^{\mathrm{eff}}(G)$ it represents in $\mathbf{DM}^{\mathrm{eff}}(S)$. For this, we need a cofibrant resolution of $G \otimes \mathbb{Q}$. We use a construction of Breen [Bre70] that is based on the work of Eilenberg and Maclane [EML53].

The following theorem is the main result of this appendix.

Theorem 2.A.1. *Let $(\mathcal{S}, \tau)$ be a Grothendieck site. We denote $\mathbb{Z}(-)$ the functor "free abelian group sheaf" (the sheafification of the sectionwise free abelian group functor).*

There is a functor:

$$A : \mathbf{Sh}_\tau(S, \mathbb{Z}) \to \mathbf{Cpl}_{\geq 0}\, \mathbf{Sh}_\tau(S, \mathbb{Z})$$

(where by $\mathbf{Cpl}_{\geq 0}$ we mean homological complexes in non-negative degrees) together with a natural transformation

$$r : A \to (-)[0]$$

satisfying the following properties:

1. *For all $\mathcal{G} \in \mathbf{Sh}_\tau(S, \mathbb{Z})$ and $i \geq 0$, the sheaf $A(\mathcal{G})_i$ is of the form $\bigoplus_{j=0}^{d(i)} \mathbb{Z}(\mathcal{G}^{a(i,j)})$ for some $d(i), a(i,j) \in \mathbb{N}$.*
2. *There is a natural transformation $\tilde{a} : \mathbb{Z}(-)[0] \to A$ which lifts the addition map $a : \mathbb{Z}(-) \to \mathrm{id}$; that is, one has $a[0] = r\tilde{a}$.*
3. *The functor A and the transformations r and $\tilde{a}$ are compatible with pullbacks by morphisms of sites.*
4. *The map $r \otimes \mathbb{Q}$ is a quasi-isomorphism.*

We give a sketch of the proof of Theorem 2.A.1 based on [Bre70] and [EML53].

Construction of $A(\mathcal{G})$

The construction of the chain complex $A(\mathcal{G})$ is done in four steps.

(i) The chain complex $A(\mathcal{G}, 0)$ is defined as $\mathbb{Z}(\mathcal{G})[0]$, equipped with its natural ring structure.

(ii) To define recursively $A(\mathcal{G}, n+1)$ from $A(\mathcal{G}, n)$, one applies the *normalized bar construction* B_N defined in [EML53, Chapter II] (see also Remark 2.A.3)

$$A(\mathcal{G}, n+1) = B_N(A(\mathcal{G}, n)).$$

(iii) For all $n \geq 1$, we define $A^n(G)$ by taking the canonical truncation (intuitively passing to the reduced homology) and shifting:

$$A^n(\mathcal{G}) = (\tau_{>0} A(\mathcal{G}, n))[-n]\ .$$

(iv) The bar construction comes together with a functorial suspension map

$$S : A(\mathcal{G}, n)[1] \to A(\mathcal{G}, n+1) \ .$$

For all $n \geq 1$, up to a change of sign in the differential, the suspension S induces a morphism of complexes

$$S : A^n(\mathcal{G}) \to A^{n+1}(\mathcal{G}) \ .$$

The maps are well-defined by the vanishing properties in Lemma 2.A.2 below, and they are inclusions. Define the chain complex $A(\mathcal{G})$ as

$$A(\mathcal{G}) = \bigcup_{n\geq 1} A^n(\mathcal{G}) \ .$$

Lemma 2.A.2 ([Bre70])**.** *Let $A(\mathcal{G}, n)_j$ be the j-th term of the chain complex $A(\mathcal{G}, n)$; then*

$$A(\mathcal{G}, n)_j = \begin{cases} \mathbb{Z} & n \geq 1, j = 0, \\ 0 & n \geq 1, 0 < j < n. \\ \mathbb{Z}(G) & n \geq 1, j = n \end{cases}$$

Moreover, the map $A(G, 1)_1 \to A(G, 1)_0$ is zero. In particular, for all $n \geq 1$ the chain complex $A^n(\mathcal{G})$ is concentrated in non-negative degrees and can also be described via the stupid truncation

$$A^n(\mathcal{G}) = (\sigma_{>0} A(\mathcal{G}, n))[-n] \ ,$$

which is still concentrated in non-negative degrees.

Remark 2.A.3. Recall that the bar construction B_N takes an augmented differential graded chain algebra and produces another; intuitively, if D is an augmented dg-algebra computing the homology of a space X, the dg-algebra $B_N(D)$ should be thought of as an algebraic model for the homology of the loop space of X. Note that since we work with sheaves, the correct construction is to apply the bar construction sectionwise and then sheafify.

Construction of r and end of the proof

We first recall a qualitative version of Cartan's computation of the stable homology of $A(\mathcal{G}, n)$, as extended to the sheaf case by Breen (see [Bre70, Theorem 3]):

Lemma 2.A.4. *Let $n \geq 1$. Then:*

$$\mathcal{H}_i(A^n(\mathcal{G})) \simeq \begin{cases} 0 & i < 0, \\ \mathcal{G} & i = 0, n \geq 1, \\ \textit{is a torsion sheaf} & 0 < i < n. \end{cases} \tag{2.A.0.1}$$

Moreover, the isomorphisms $\mathcal{H}_0(A^n(\mathcal{G})) \simeq \mathcal{G}$ for $n \geq 1$ can be chosen such that

1. *they are compatible with the suspension maps, and*
2. *the composition of maps of complexes*

$$\mathbb{Z}(\mathcal{G})[0] \to A^1(\mathcal{G}) \to \mathcal{H}_0(A^1(\mathcal{G})) \simeq \mathcal{G}[0]$$

is the addition map $a_{\mathcal{G}}$ (here the map $\mathbb{Z}(\mathcal{G})[0] \to A^1(\mathcal{G})$ is the inclusion of the term in degree 0).

Proof. The computation of $\mathcal{H}_i(A(\mathcal{G}, n))$ is contained in [Bre70, Theorem 3]. It reads

$$\mathcal{H}_i(A(\mathcal{G}, n)) \simeq \begin{cases} 0 & 0 < i < n, \\ \mathcal{G}, & i = n, \\ \text{is a torsion sheaf}, & n < i < 2n, \end{cases} \tag{2.A.0.2}$$

The computation of $\mathcal{H}_i(A^n(\mathcal{G}))$ follows from this and the vanishing properties of Lemma 2.A.2. Compatibility with suspensions is a consequence of [Bre70, (eq 1.16) p.22]. The second assertion follows by a small computation from the following description of the first terms of $A(\mathcal{G},1)$ (see [Bre70, p.19]):

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbb{Z}(\mathcal{G}\times\mathcal{G}) & \longrightarrow & \mathbb{Z}(\mathcal{G}) & \xrightarrow{0} & \mathbb{Z} \\ & & [g,h] & \longmapsto & [g]+[h]-[g+h] & & \end{array}$$

This finishes the proof of the Lemma. ☐

Définition 2.A.5. We define the map

$$r : A(\mathcal{G}) = \bigcup_{n\geq 1} A^n(\mathcal{G}) \to \mathcal{G}$$

via the compatible system of

$$A^n(\mathcal{G}) \to \mathcal{H}_0(A^n(\mathcal{G})) \cong \mathcal{G}$$

of Lemma 2.A.4.

Proof of the Theorem 2.A.1. Property (1) follows from [Bre70, eq. (3.8) and (3.9) p.31].

The morphism r satisfies property (2) by construction and Lemma 2.A.4. It induces an isomorphism $\mathcal{H}_0(r) : \mathcal{H}_0(A(\mathcal{G})) \simeq \mathcal{G}$. The construction is sufficiently canonical to make property (3) clear.

Finally as the computation of homology commutes with filtered colimits and $\mathbb{Q}$ is flat over $\mathbb{Z}$, we have for all $i > 0$

$$\mathcal{H}_i(A(\mathcal{G})) \otimes \mathbb{Q} \simeq \mathrm{colim}_{n\in\mathbb{N}}(\mathcal{H}_i(A^n(\mathcal{G})) \otimes \mathbb{Q}).$$

By Equation 2.A.0.1, the last term vanishes for $n > i$. This proves property (4) and concludes the proof. ☐

2.B qfh-descent for smooth group schemes

Let S be a noetherian excellent scheme. In this section we study the qfh sheafification of smooth commutative group schemes over S.

Remark 2.B.1. Let **Nor** be the full subcategory of S-schemes of finite type which are normal. Let T be an S-scheme of finite type. Then there is a normal S-scheme T' and a finite surjective morphism (hence a qfh cover) $T' \to T$: take the normalisation of the reduction of the union of irreducible components of T. In particular the subsite **Nor** equipped with the induced qfh topology is dense in Sch, and by [SGA72, Exposé III Théorème 4.1] the inclusion induces an equivalence of topoi.

Let G/S a smooth group scheme and G be the étale sheaf on Sch/S defined by G. The main result of this appendix is the following.

Proposition 2.B.2. *Let G be a smooth commutative group scheme over S. Then $G_{\mathbb{Q}}|_{\mathbf{Nor}}$ is a qfh sheaf, and the natural morphism*

$$G_{\mathbb{Q}} \to (G_{\mathbb{Q}})_{\mathrm{qfh}}$$

induces isomorphisms when evaluated on normal schemes.

The second statement is a reformulation of the first by Remark 2.B.1. The proof of the fact that $G_{\mathbb{Q}}|_{\mathbf{Nor}}$ is a qfh sheaf will take the rest of this appendix.

Lemma 2.B.3. *Let $f : X \to Y$ be a dominant morphism with Y reduced. Then $G(Y) \to G(X)$ is injective. In particular the presheaf $G|_{\mathbf{Nor}}$ is separated with respect to the* qfh*-topology.*

Proof. The morphism f is schematically dominant since Y is reduced [Gro66b, Proposition 11.10.4]. Let $g : Y \to G$ such that $g \circ f = 0 \in G(X)$. Since the scheme G is separated, we can now apply [Gro66b, Proposition 11.10.1 d)] to g and the constant zero morphism $X \to G$ which shows that $g = 0 \in G(Y)$. The qfh-separation property then follows from [Voe96, Proposition 3.1.4]. ☐

Lemma 2.B.4. *Let $\pi : U \to X$ be a finite surjective morphism of normal irreducible schemes. Then $G_{\mathbb{Q}}$ satisfies the sheaf condition for the qfh cover π.*

Proof. Let V be the normalisation of the reduction of $U \times_X U$. We have to show that the sequence

$$0 \to G(X) \otimes \mathbb{Q} \to G(U) \otimes \mathbb{Q} \to G(V) \otimes \mathbb{Q}$$

is exact (the map $G(U) \otimes \mathbb{Q} \to G(V) \otimes \mathbb{Q}$ being induced by the difference of the two projections $V \to U$).

As X and U are normal, there exists $\pi' : Y \to U$ finite surjective such that $\pi'\pi$ factors as $\pi_s \pi_i$ with π_s, π_i finite surjective, π_s generically Galois and π_i generically purely inseparable. Because of Lemma 2.B.3, a diagram chase shows that the sheaf condition for π is implied by the sheaf condition for π_s and π_i separately. In other words we can assume π to be either generically Galois or generically purely inseparable.

We first assume π to be generically Galois with Galois group Γ. Note that U, being normal, is the normalisation of X in $\kappa(U)$ so the action of Γ on the generic fibre extends to U by functoriality of the normalisation. Moreover, the monomorphism of coherent sheaves $\mathcal{O}_X \to (\pi_* \mathcal{O}_Y)^\Gamma$ is an isomorphism, as can be checked on affine charts using the generic Galois property and normality. Hence by [SGA03, Exposé V Proposition 1.3], X is the categorical quotient of U by Γ. In particular we have:

$$G(X) = G(U)^\Gamma \subset G(U) \ .$$

Let x be in the kernel of $G(U) \to G(V)$. Then by Lemma 2.B.3 it is also in the kernel of $G(k(U)) \to G(k(U) \otimes_{k(X)} k(U))$ (recall that $k(U) \otimes_{k(X)} k(U)$ is a product of fields, hence normal). By étale descent we have $x \in G(k(U))^\Gamma$, in particular, our element is Γ-invariant, hence in $G(X)$ by the above.

Consider now the case π generically inseparable. By Zariski's connectedness theorem, a finite surjective morphism to a normal scheme which is generically purely inseparable is purely inseparable. In particular the diagonal morphism of π is a surjective closed immersion, hence after reduction is an isomorphism. Hence the map $G(U) \otimes \mathbb{Q} \to G(V) \otimes \mathbb{Q}$ is the zero map (the identity minus the identity) and we have to show that $\pi^* \otimes \mathbb{Q} : G(X) \otimes \mathbb{Q} \to G(U) \otimes \mathbb{Q}$ is surjective. For this we will reduce to the case where π is a relative Frobenius.

Let us recall some notations. For p prime, q a prime power and an S-scheme Z, we write $Z^{(q)}$ for the base change of Z along the absolute Frobenius of S, and $\mathrm{Frob}^{(q)}_{Z/S} : Z \to Z^{(q)}$ for the relative Frobenius.

We can assume that π is not an isomorphism. Let then $p > 0$ be the generic characteristic of X. Since X is irreducible, it is an $\mathbb{F}_p$-scheme. By [Kol97, Proposition 6.6], there exists a power q of p and a morphism $\pi^{-q} : X \to U^{(q)}$ such that $\pi^{-q}\pi = \mathrm{Frob}^{(q)}_{U/S}$. Hence we are reduced to the Frobenius case as surjectivity of $(\mathrm{Frob}^{(q)}_{U/S})^* \otimes \mathbb{Q}$ implies surjectivity of $\pi^* \otimes \mathbb{Q}$.

Let $f \in G(U)$. By functoriality of Frobenius, we get a morphism

$$f^{(q)} \in G^{(q)}(U^{(q)})$$

such that $f^{(q)}\, \mathrm{Frob}^{(q)}_{U/S} = \mathrm{Frob}^{(q)}_{G/S}\, f$. The commutative S-group scheme G is flat, so by [ABD$^+$65, Exposé VII 4.3] there exists a Verschiebung morphism $\mathrm{Ver}^{(q)}_{G/S} : G^{(q)} \to G$ such that $\mathrm{Ver}^{(q)}_{G/S} \circ \mathrm{Frob}^{(q)}_{G/S} = q\mathrm{id}_G$. Put

$$g = \mathrm{Ver}^{(q)}_{G/S} f^{(q)} \in G(U^{(q)}) \ .$$

Then $\pi^*(g) = pf$. We conclude that $\pi^* \otimes \mathbb{Q}$ is surjective. □

Proof of Proposition 2.B.2. Let $X \in \mathbf{Nor}$ and $\{p_i : Y_i \to X\}_{i \in I}$ a qfh-cover in $\mathbf{Nor}$. We have to check the sheaf condition for $G_{\mathbb{Q}}$.

The presheaf $G_{\mathbb{Q}}$ satisfyies the sheaf condition for the covering of a scheme by the union of its connected components so that we can assume X to be connected, hence integral. By [SV96, Lemma 10.3], the cover $\{p_i\}$ admits a refinement to a cover $\{Z_i \to Z \to X\}_{j \in J}$ where $Z \to X$ is finite surjective and $\{Z_i \to Z\}$ is a Zariski cover of Z. Because of Lemma 2.B.3, a diagram chase shows that it is implied by the sheaf condition for $Z \to X$ and $\{Z_i \to Z\}$ separately.

By Lemma 2.B.4, the presheaf $G_{\mathbb{Q}}$ satisfies the sheaf condition for the morphism $Z \to X$. As $G_{\mathbb{Q}}$ is a Zariski-sheaf, the sheaf condition is also satisfied for the cover $\{Z_i\}$ of Z. This concludes the proof. $\square$

Chapter 3

Relative 1-motives

3.1 Triangulated categories of n-motives

Categories of motives are naturally filtered by the dimension of "geometric generators", and such filtrations have been studied in various motivic contexts [Gro] [Ayo11] [ABV09]. We give definitions in the context of $\mathbf{DA}(-)$ and prove a number of basic results. Since such a treatment does not appear in the litterature, we provide more than is strictly necessary for the rest of the paper; outside of this section, we are concerned with the special case of (co)homological 0- and 1-motives.

3.1.1 Definitions

We fix a base scheme S and an integer $n \geq 0$ for the remainder of this section.

Definition 3.1.1. The category $\mathbf{DA}^{\mathrm{coh}}(S)$ (resp. $\mathbf{DA}_{\mathrm{hom}}(S)$) of *cohomological motives* (resp. *homological motives*) is the full subcategory of $\mathbf{DA}(S)$ defined as

$$\mathbf{DA}^{\mathrm{coh}}(S) = \ll f_*\mathbb{Q}_X |\ f : X \to S \text{ proper morphism} \gg$$

(resp.

$$\mathbf{DA}_{\mathrm{hom}}(S) = \ll f_\sharp\mathbb{Q}_X |\ f : X \to S \text{ smooth morphism} \gg).$$

The category $\mathbf{DA}^n(S)$ (resp. $\mathbf{DA}_n(S)$) of *cohomological n-motives* (resp. *homological n-motives*) is the full subcategory of $\mathbf{DA}(S)$ defined as

$$\mathbf{DA}^n(S) = \ll f_*\mathbb{Q}_X |\ f : X \to S \text{ proper morphism of relative dimension} \leq n \gg$$

(resp.

$$\mathbf{DA}_n(S) = \ll f_\sharp\mathbb{Q}_X |\ f : X \to S \text{ smooth morphism of relative dimension} \leq n \gg).$$

Remark 3.1.2. As we will see in Proposition 3.1.28, the categories $\mathbf{DA}_n(S)$ and $\mathbf{DA}^n(S)$ are in fact equivalent as triangulated categories, so that many questions about $\mathbf{DA}^n(S)$ can be reduced to $\mathbf{DA}_n(S)$. In the special cases $n = 0, 1$, this is a crucial ingredient for several results in this paper. However to establish Proposition 3.1.28 we need some preliminary results which we obtain by studying $\mathbf{DA}_n$ and $\mathbf{DA}^n$ in parallel.

We have subcategories of smooth and geometrically smooth objects. Recall that an object X in a symmetric monoidal category is said to be strongly dualizable if there exists an object X^* together with morphisms $\epsilon : \mathbb{1} \to X \otimes X^*$ and $\eta : X \otimes X^* \to \mathbb{1}$ satisfying the classical adjunction triangle laws.

Definition 3.1.3. The category $\mathbf{DA}^{\mathrm{gsm}}(S)$ (resp. $\mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm}}(S)$, $\mathbf{DA}^{\mathrm{gsm}}_{\mathrm{hom}}(S)$) of *geometrically smooth motives* (resp. *geometrically smooth cohomological motives* resp. of *geometrically smooth homological motives*) is the full subcategory of $\mathbf{DA}(S)$ defined as

$$\mathbf{DA}^{\mathrm{gsm}}(S) = \ll f_\sharp\mathbb{Q}_X(-n) |\ f : X \to S \text{ proper smooth morphism},\ n \in \mathbb{Z} \gg$$

(resp.
$$\mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm}}(S) = \ll f_*\mathbb{Q}_X|\ f : X \to S \text{ proper smooth morphism} \gg,$$
$$\mathbf{DA}^{\mathrm{gsm}}_{\mathrm{hom}}(S) = \ll f_\sharp\mathbb{Q}_X|\ f : X \to S \text{ proper smooth morphism} \gg).$$

The category $\mathbf{DA}^{\mathrm{sm}}_S$ (resp, $\mathbf{DA}^{\mathrm{coh}}_{\mathrm{sm}}(S)$, $\mathbf{DA}^{\mathrm{sm}}_{\mathrm{hom}}(S)$) of *smooth motives* (resp. *smooth cohomological motives*, *smooth homological motives*) is defined as

$$\mathbf{DA}^{\mathrm{coh}}_{\mathrm{sm}}(S) = \ll M \in \mathbf{DA}(S)|\ M \text{ strongly dualizable } \gg$$

(resp.
$$\mathbf{DA}^{\mathrm{coh}}_{\mathrm{sm}}(S) = \ll M \in \mathbf{DA}^{\mathrm{coh}}(S)|\ M \text{ strongly dualizable in } \mathbf{DA}(S) \gg,$$
$$\mathbf{DA}^{\mathrm{sm}}_{\mathrm{hom}}(S) = \ll M \in \mathbf{DA}_{\mathrm{hom}}(S)|\ M \text{ strongly dualizable in } \mathbf{DA}(S) \gg).$$

We then define $\mathbf{DA}^n_{\mathrm{gsm}}(S)$ as $\mathbf{DA}^n(S) \cap \mathbf{DA}_{\mathrm{gsm}}(S)$, etc.

Remark 3.1.4. In the definition of $\mathbf{DA}^n_{\mathrm{gsm}}(S)$ and $\mathbf{DA}^{\mathrm{gsm}}_n(S)$, we make the choice not to impose the geometric smoothness to "come from" generators of relative dimension $\leq n$. This more restrictive definition would be too strong for our purpose of formulating a reasonable conjecture on geometrically smooth 1-motives, see Corollary 3.2.16 and Conjecture 3.5.7 below.

Lemma 3.1.5. *Geometrically smooth objects are smooth:* $\mathbf{DA}^{\mathrm{gsm}}(S) \subset \mathbf{DA}^{\mathrm{sm}}(S)$, $\mathbf{DA}^{\mathrm{gsm}}_{\mathrm{hom}}(S) \subset \mathbf{DA}^{\mathrm{sm}}_{\mathrm{hom}}(S)$*, etc.*

Proof. This follows from relative purity and the projection formula, see e.g. [CDa, Lemma 4.2.8]. □

Remark 3.1.6. The converse of the above lemma is not known and it is not clear if one should expect it. Informally, when S is a discrete valuation ring, it would mean that a "motive with good reduction" is realisable in the cohomology of a variety with good reduction.

There is a further reasonable definition of a smooth object in $\mathbf{DA}_c(S)$, namely a motive whose realisations have cohomology sheaves which are local systems (in the appropriate sense, e.g. lisse ℓ-adic sheaves). This is conjecturally equivalent to being strongly dualizable.

Proposition 3.1.26 shows that when S is the spectrum of a field, any motive is geometrically smooth.

An important property of smooth compact objects is that they satisfy a form of absolute purity.

Proposition 3.1.7. *Let $i : Z \to S$ be a regular immersion of codimension c. For $M \in \mathbf{DA}^{\mathrm{sm}}_c(S)$ (i.e., M strongly dualisable), there is a purity isomorphism*

$$i^*M \simeq i^!M(c)[2c]$$

which is functorial in M, in the sense that for any $f : M \to N \in \mathbf{DA}^{\mathrm{sm}}_c(S)$ the diagram

$$\begin{tikzcd} i^*M \arrow[r, "i^*(f)"'] \arrow[d] & i^*N \arrow[d] \\ i^!M(c)[2c] \arrow[r, "i^!(f)(c)[2c]"'] & i^!N(c)[2c] \end{tikzcd}$$

commutes.

Proof. The idea is to use dualisability to reduce to the usual absolute purity property for the unit object. The functor i^* is monoidal, hence preserves strongly dualisable objects and sends strong duals to strong duals. By biduality, this provides a natural isomorphism

$$i^*M \xrightarrow{\sim} \underline{\mathsf{Hom}}(i^*M^\vee, \mathbb{Q}_Z)$$

By absolute purity, this last group is isomorphic to $\underline{\mathsf{Hom}}(i^*M^\vee, i^!\mathbb{Q}_Z(c)[2c])$, which is itself naturally isomorphic to $i^!\underline{\mathsf{Hom}}(M^\vee, \mathbb{Q}(c)[2c])$ by [Ayo07a, Proposition 2.3.55]. Since $\mathbb{Q}(c)[2c]$ is invertible and M is strongly dualisable, $i^!\underline{\mathsf{Hom}}(M^\vee, \mathbb{Q}(c)[2c]) \simeq i^!\underline{\mathsf{Hom}}(M^\vee, \mathbb{Q})(c)[2c] \simeq i^!M(c)[2c]$. The composition gives the required isomorphism. Each step of the construction is functorial in M. □

Lemma 3.1.8. *Let $\mathcal{T}$ be one of $\mathbf{DA}_{\mathrm{hom}}(S)$, $\mathbf{DA}^{\mathrm{coh}}(S)$, $\mathbf{DA}_n(S)$, $\mathbf{DA}^n(S)$ or their subcategories of smooth or geometrically smooth objects. Then the triangulated category $\mathcal{T}$ is compactly generated by its generating family, and an object of $\mathcal{T}$ is compact if and only if it is compact in $\mathbf{DA}(S)$.*

Proof. Write $\mathcal{G}$ for the generating family of $\mathcal{T}$. By [Ayo14a, Proposition 3.20, Proposition 8.5] and by the fact that strongly dualizable objects in a symmetric monoidal triangulated category are automatically compact, we see that all objects of $\mathcal{G}$ are compact. This means that $\mathcal{T}$ is compactly generated by $\mathcal{G}$. Write $\mathcal{T}_c$ for the full subcategory of objects of $\mathcal{T}$ which are compact in $\mathcal{T}$. By [Nee01, Lemma 4.4.5], $\mathcal{T}_c = \langle \mathcal{G} \rangle$. In particular any object of $\mathcal{T}_c$ is compact in $\mathbf{DA}(S)$. The converse implication is obvious. $\square$

Definition 3.1.9. We write $\mathbf{DA}^{\mathrm{coh}}_c(S)$, $\mathbf{DA}_{\mathrm{hom},c}(S)$, etc. for the full subcategories of compact objects of $\mathbf{DA}^{\mathrm{coh}}(S)$, $\mathbf{DA}_{\mathrm{hom}}(S)$, etc.

3.1.2 Permanence properties

The subcategories we have introduced are each stable under a specific subset of Grothendieck operations. We start with the compatibilities with the monoidal structure.

Proposition 3.1.10. *Let S be a base scheme.*

(i) *$\mathbf{DA}^{\mathrm{coh}}(S)$ is stable by tensor products and negative Tate twists.*

(ii) *For all $m, n \geq 0$, we have $\mathbf{DA}^m(S) \otimes \mathbf{DA}^n(S) \subset \mathbf{DA}^{m+n}(S)$.*

(iii) *For all $m, n \geq 0$, we have $\mathbf{DA}^m(S)(-n) \subset \mathbf{DA}^{m+n}(S)$.*

(iv) *$\mathbf{DA}_{\mathrm{hom}}(S)$ is stable by tensor products and positive Tate twists.*

(v) *For all $m, n \geq 0$, we have $\mathbf{DA}_m(S) \otimes \mathbf{DA}_n(S) \subset \mathbf{DA}_{m+n}(S)$.*

(vi) *For all $m, n \geq 0$, we have $\mathbf{DA}_m(S)(n) \subset \mathbf{DA}_{m+n}(S)$.*

The same properties hold for the smooth and geometrically smooth versions.

Proof. First, note that $\otimes$ commutes with small sums in both variables, being a left adjoint. This reduces the proof to checking the result for generators.

Let us prove point (i). Recall that we have a projection formula for $f_!$ and f^* from [Ayo07a, Theoreme 2.3.40], i.e., for any finite type separated morphism $f : S \to T$ and any $M \in \mathbf{DA}(S), N \in \mathbf{DA}(T)$, we have a natural isomorphism

$$f_!(M \otimes f^*N) \simeq f_!M \otimes N.$$

Let $g : X \to S$ and $h : Y \to S$ be proper morphisms. Let $Z = X \times_S Y$ and let $g' : Z \to Y$ and $h' : Z \to X$ be the two projections. We have a sequence of isomorphisms

$$\begin{aligned} g_*\mathbb{Q}_X \otimes h_*\mathbb{Q}_Y &\simeq g_!\mathbb{Q}_X \otimes h_!\mathbb{Q}_Y \\ &\simeq g_!(\mathbb{Q}_X \otimes g^*h_!\mathbb{Q}_Y) \\ &\simeq g_!h'_!(g')^*\mathbb{Q}_Y \\ &\simeq g_*h'_*\mathbb{Q}_Z \end{aligned}$$

where the first and fourth isomorphisms follows from properness, the second is the projection formula and the third is the $\mathrm{Ex}^*_!$ isomorphism. This shows that $g_*\mathbb{Q}_X \otimes h_*\mathbb{Q}_Y$ is cohomological. The negative Tate twist $\mathbb{Q}_S(-n)$ is cohomological, as it is a direct factor of $(\mathbb{P}^n_S \to S)_*\mathbb{Q}$. This finishes the proof of (i). The same proof, combined with the fact that relative dimension is stable by base change and adds up in compositions, gives (ii) and (iii).

For the proof of point (iv), we use a parallel argument; we combine the projection formula for $f_\sharp$ and f^* of [Ayo07b, Proposition 4.5.17] with the $\mathrm{Ex}^*_\sharp$ isomorphism and the fact that $\mathbb{Q}_S(n)$ is a direct factor of $(\mathbb{P}^n_S \to S)_\sharp\mathbb{Q}$ by the projective bundle formula. The same proof, combined with the fact that relative dimension is stable by base change and adds up in compositions, gives (v) and (vi). $\square$

Proposition 3.1.11. *Let $f : S \to T$ be a morphism of schemes. The following operations preserve the subcategories $\mathbf{DA}^{\mathrm{coh}}(-)$.*

(i) *f^* for any f.*

(ii) *f_* when f is separated of finite type and S admits the resolution of singularities by alterations.*

(iii) *$f_!$ when f is separated of finite type.*

(iv) *$f^!$ when f is quasi-finite separated and S admits the resolution of singularities by alterations.*

Moreover, they also preserve $\mathbf{DA}_c^{\mathrm{coh}}(-)$ (with the assumption that the schemes involved are excellent for points (ii)-(iv)*).*

Proof. The results for $\mathbf{DA}_c^{\mathrm{coh}}(-)$ follow from the ones for $\mathbf{DA}^{\mathrm{coh}}(-)$ together with the constructibility theorem [Ayo14a, Theoreme 8.10] and Lemma 3.1.8. We thus focus on $\mathbf{DA}^{\mathrm{coh}}(-)$. We prove the results in a slightly different order than in the statement: we first establish (i), (iii) (which contains the special case of (ii) for proper morphisms) , (iv) for closed immersions, (ii) and finally (iv) in all generality. In each case, we first check that the functor commutes with small sums, and then compute its action on generators of $\mathbf{DA}^{\mathrm{coh}}(-)$.

Proof of (i)*:* the functor f^* is a left adjoint hence commutes with small sums. Moreover proper base change implies that f^* sends generators of $\mathbf{DA}^{\mathrm{coh}}(T)$ to generators of $\mathbf{DA}^{\mathrm{coh}}(S)$.

Proof of (iii)*:* the functor $f_!$ is a left adjoint hence commutes with small sums. Let $g : X \to S$ be a proper morphism. We need to show that $f_! g_* \mathbb{Q}_X \simeq (f \circ g)_! \mathbb{Q}_X$ is in $\mathbf{DA}^{\mathrm{coh}}(T)$. Because f is assumed to be separated of finite type, the same holds for $f \circ g$. Nagata's theorem [Nag63] [Con07] implies that $f \circ g$ admits a compactification, i.e., that there exists a factorisation $f \circ g = \bar{f} \circ j$ with $j : X \to \bar{X}$ an open immersion and $\bar{f} : \bar{X} \to T$ a proper morphism. Let $i : Z \to \bar{X}$ be a complementary closed immersion to j. By localisation, we have a distinguished triangle

$$j_! \mathbb{Q}_X \to \mathbb{Q}_{\bar{X}} \to i_! \mathbb{Q}_Z \xrightarrow{+}$$

which after applying $\bar{f}_* \simeq \bar{f}_!$ yields

$$\bar{f}_* j_! \mathbb{Q}_X \simeq f_! g_* \mathbb{Q}_X \to \bar{f}_! \mathbb{Q}_{\bar{X}} \to (\bar{f} i)_! \mathbb{Q}_Z \xrightarrow{+} .$$

By definition, the second and third terms in this triangle are in $\mathbf{DA}^{\mathrm{coh}}(T)$. This implies that the first is as well.

Proof of (iv) *for $f = i$ closed immersion:*

The functor $i^!$ has a left adjoint $i_!$ which sends compact objects to compact objects by [Ayo14a, Proposition 8.5]. By [Ayo07a, Lemme 2.1.28] this implies that $i^!$ commutes with small sums.

The blueprint for this proof is taken from Section 2.2.2 of [Ayo07a]. Before we start, we need a lemma providing convenient generators for $\mathbf{DA}^{\mathrm{coh}}$.

Lemma 3.1.12. *Let S be a scheme admitting resolution of singularities by alterations, $f : X \to S$ a finite type morphism and $T \subset X$ closed. Then $\mathbf{DA}^{\mathrm{coh}}(X)$ is compactly generated by motives of the form $g_* \mathbb{Q}_{X'}$ with $g : X' \to X$ a projective morphism and X' connected regular and $g^{-1}(T)$ equal either to X' or to a normal crossing divisor.*

Proof. The reference [Ayo07a, Proposition 2.2.27], specialized to the $\mathbb{Q}$-linear, separated, homotopical 2-functor $\mathbf{DA}(-)$ proves a similar statement for the category of constructible objects $\mathbf{DA}_c(S)$ (with added positive Tate twists of the generators, and restriction to quasi-projective morphisms). Once one removes the Tate twists, the restriction to quasi-projective morphisms, and remarks that Statement (iii) which we just established provides the analogue of Corollaire 2.2.21 from loc.cit, the same argument applies verbatim. $\square$

Lemma 3.1.13, applied to $i : S \to T$, shows that it is enough to that for $i^! g_* \mathbb{Q}_X$ for any $g : X \to T$ with X connected regular and $g^{-1}(S)$ equal to either X or a normal crossing divisor.

Form the cartesian square

$$\begin{tikzcd} Y \arrow[r, "i'"] \arrow[d, "g'"'] & X \arrow[d, "g"] \\ S \arrow[r, "i"'] & T. \end{tikzcd}$$

We have an $\mathrm{Ex}^!_*$ isomorphism $i^! g_* \mathbb{Q}_X \simeq g'_* i'^! \mathbb{Q}_X$. By point (iii), it is enough to show that $i'^! \mathbb{Q}_X$ is in $\mathbf{DA}^{\mathrm{coh}}(X)$. Only the case of a normal crossing divisor requires a proof. By [Ayo07a, Lemme 2.2.31] applied to the branches and point (iii) for closed immersions, we reduce to the case of a regular immersion, which then follows from absolute purity and Proposition 3.1.10 (i).

Proof of (ii)*:*

Using Nagata's theorem and the proper case of point (iii), we reduce to show that $j_* \mathbb{Q}_S$ is in $\mathbf{DA}^{\mathrm{coh}}(T)$ for $j : S \to T$ open immersion. This now follows from colocalisation and point (iv) for the complementary closed immersion.

Proof of (iv) *for f quasi-finite general:*

By the same argument as above, using the $\mathrm{Ex}^!_*$ isomorphism, it is enough to show that $f^! \mathbb{Q}_T$ is in $\mathbf{DA}^{\mathrm{coh}}(S)$. Using Zariski's main theorem [Gro66b, Théorème 8.12.6], the fact that $j^! \simeq j^*$ for j open immersion, and by point (i) we are reduced to the case of finite morphisms.

If f is finite étale, then $f^! \simeq f^*$ again and we are done by point (i). If f is finite and purely inseparable, then a corollary of the separation property of $\mathbf{DA}$ is that $f^! \simeq f^* \simeq$ is an equivalence of categories [Ayo07a, Corollaire 2.1.164]. In general, we proceed by induction on the dimension of T. Generically on T, say above an everywhere dense open set $j : U \to T$, f is the composite of a finite étale morphism (possibly empty) followed by a finite purely inseparable morphism. Let $l : V \to S$ be $j \times_T S$ and $k : W \to S$ be a complementary closed immersion. Then $l^! f^! \mathbb{Q}_T \simeq f^!_U \mathbb{Q}_U$ is in $\mathbf{DA}^{\mathrm{coh}}(V)$ by the arguments above. By point (ii), we get that $l_* l^* f^! \mathbb{Q}_T$ is in $\mathbf{DA}^{\mathrm{coh}}(S)$. By induction on the dimension and point (iii), we get that $k_! k^! f^! \mathbb{Q}_T$ lies in $\mathbf{DA}^{\mathrm{coh}}(S)$. The colocalisation triangle then shows that $f^! \mathbb{Q}_T$ lies in $\mathbf{DA}^{\mathrm{coh}}(S)$ and this concludes the proof. □

Lemma 3.1.13. *Let S be a scheme admitting resolution of singularities by alterations, $f : X \to S$ a finite type morphism and $T \subset X$ closed. Then $\mathbf{DA}^{\mathrm{coh}}(X)$ is compactly generated by motives of the form $g_* \mathbb{Q}_{X'}$ with $g : X' \to X$ a projective morphism and X' connected regular and $g^{-1}(T)$ equal either to X' or to a normal crossing divisor.*

Proof. The reference [Ayo07a, Proposition 2.2.27], specialized to the $\mathbb{Q}$-linear, separated, homotopical 2-functor $\mathbf{DA}(-)$ proves a similar statement for the category of constructible objects $\mathbf{DA}_c(S)$ (with added positive Tate twists of the generators, and restriction to quasi-projective morphisms). Once one removes the Tate twists, the restriction to quasi-projective morphisms, and remarks that Statement Proposition (iii) provides the analogue of Corollaire 2.2.21 from loc.cit, the same argument applies verbatim. □

Proposition 3.1.14. *Let $f : S \to T$ be a morphism of schemes. The following operations preserve the subcategories $\mathbf{DA}_{\mathrm{hom}}(-)$ and $\mathbf{DA}_{\mathrm{hom},c}(-)$.*

(i) *f^* for any f.*

(ii) *$f_\sharp$ when f is smooth.*

(iii) *$f^!$ when f is smooth.*

(iv) *$f_!$ for any quasi-finite separated morphism f.*

Remark 3.1.15. In the proof of point (iv), we use results from Sections 3.1.3 and 3.1.4. The careful reader can check that we do not use the reference 3.1.14 (iv) in between. We feel this break from logical order is justified by the commodity of having a clean statement.

Proof. The results about $\mathbf{DA}_{\mathrm{hom},c}(-)$ follow from the ones for $\mathbf{DA}_{\mathrm{hom}}(-)$ together with the constructibility result [Ayo14a, Proposition 8.5] and Lemma 3.1.8. We thus focus on $\mathbf{DA}_{\mathrm{hom}}(-)$.

Proof of (i)*:* The functor f^* is a left adjoint so commutes with small sums. Moreover the $\mathrm{Ex}^*_\sharp$ isomorphism implies that f^* sends generators of $\mathbf{DA}_{\mathrm{hom}}(T)$ to generators of $\mathbf{DA}_{\mathrm{hom}}(S)$.

Proof of (ii)*:* The functor $f_\sharp$ is a left adjoint so commutes with small sums. The fact that generators are sent to homological motives clearly follows from the definition.

Proof of (iii)*:* This follows from relative purity together with (i) and 3.1.10 (vi).

Proof of (iv)*:* The functor $f_!$ is a left adjoint so preserves small sums. Using Zariski's Main theorem [Gro66b, Théorème 8.12.6] and (ii), we see that it is enough to treat the case of f finite.

We first do the case of closed immersions. The next lemma is proved using Mayer-Vietoris distinguished triangles.

Lemma 3.1.16. *Let T be a scheme and $\mathcal{U} = \{j_k : U_k \hookrightarrow T\}_{k=1}^n$ be a finite Zariski open covering of T. Let $M \in \mathbf{DA}(S)$ Then*

$$M \in \mathbf{DA}_{\mathrm{hom}}(S) \Longleftrightarrow \text{ for all } 1 \leq k \leq n, \text{ we have } j_k^* M \in \mathbf{DA}_{\mathrm{hom}}(S).$$

□

Let $i : Z \to X$ be a closed immersion and $g : U \to Z$ be a smooth morphism. We need to show that $i_* g_\sharp \mathbb{Q}_U \in \mathbf{DA}_{\mathrm{hom}}(X)$. There exists a finite open affine cover $\{U_k = \mathbf{Spec}(A_k)\}_{1 \leq k \leq n}$ of U and a finite open affine cover $\{Z_k = \mathbf{Spec}(R_k)\}_{1 \leq k \leq n}$ of Z with $g(U_k) \subset Z_k$ and such that via $g_k := g_{|U_k}^{|Z_k}$, the ring A_k takes the form:

$$A_k = R_k[x_1, \ldots, x_{n_k}]/(f_1^k, \ldots, f_{c_k}^k)$$

with $\left(\det(\frac{\partial f_j^k}{\partial x_k})\right)$ invertible in A_k (i.e. g_k is a standard smooth map). We can choose an open affine cover $\{W_k\}$ of X such that $W_k \cap Z = Z_k$. Applying Lemma 3.1.16 to the open cover W_k and using base change for closed immersions and smooth base change, we can suppose that g itself is a standard smooth map and that $X = \mathbf{Spec}(R)$ is affine.

In this situation, we can lift the equations f_j to $\tilde{f}_j \in R[x_1, \ldots, x_n]$. The open set W of X over which the resulting map $\tilde{g} : \mathbf{Spec}(R[x_1, \ldots, x_n]/(\tilde{f}_1, \ldots, \tilde{f}_n)) \to X$ is standard smooth contains Z, and $\tilde{g}$ extends g. We have a localisation triangle

$$(W \setminus Z \to W)_\sharp \tilde{g}_\sharp \mathbb{Q} \to \tilde{g}_\sharp \mathbb{Q} \to i_* g_\sharp \mathbb{Q}_U \xrightarrow{+}$$

where the first two terms are in $\mathbf{DA}_{\mathrm{hom}}(X)$. We deduce that $i_* g_\sharp \mathbb{Q}_U \in \mathbf{DA}_{\mathrm{hom}}(X)$ as wanted.

For a general quasi-finite $f : T \to S$, using localisation, the case of closed immersions and an induction on the dimension of S, we see that we can replace S by any everywhere dense open subset. The case of closed immersion also ensures we can assume S is reduced. By continuity for $\mathbf{DA}_{\mathrm{hom}}(-)$ (proven in Proposition 3.1.22 below; the proof does not use permanence properties of $\mathbf{DA}_{\mathrm{hom}}(-)$ besides (i)), we see that we can even replace S by any of its generic points. We are thus reduced to the case of a finite field extension, which follows from the following more precise Lemma 3.1.27 below. □

Proposition 3.1.17.

(i) *Let f be any morphism of schemes. Then f^* preserves the subcategories $\mathbf{DA}^n(-)$ and $\mathbf{DA}^n_c(-)$.*

(ii) *Let $f : S \to T$ be separated of finite type and of relative dimension m. Then $f_!$ sends $\mathbf{DA}^n(S)$ (resp. $\mathbf{DA}^n_c(S)$) to $\mathbf{DA}^{n+m}(T)$ (resp. $\mathbf{DA}^{n+m}_c(T)$). In particular, if f is quasi-finite, then $f_!$ preserves the subcategories $\mathbf{DA}^n(-)$ and $\mathbf{DA}^n_c(-)$.*

Proof. To treat the case of subcategories of compact objects, we combine the following arguments with Lemma 3.1.8 and the "easy" constructibility result of [Ayo14a, Proposition 8.5]. Consequently, we only treat the case of $\mathbf{DA}^n(-)$.

Statement (i) follows from the fact that f^*, being a left adjoint, commutes with small sums, proper base change and the fact that being of relative dimension $\leq n$ is stable by base change.

The proof of (ii) is the same as that of Proposition 3.1.11 (iii), keeping track of the relative dimensions involved. □

Proposition 3.1.18.

(i) *Let f be any morphism of schemes. Then f^* preserves the subcategories $\mathbf{DA}_n(-)$ and $\mathbf{DA}_{n,c}(-)$.*

(ii) *Let $f : S \to T$ be separated of finite type and of relative dimension m. Then $f_!$ sends $\mathbf{DA}_n(S)$ (resp. $\mathbf{DA}_{n,c}(S)$) to $\mathbf{DA}_{n+m}(T)$ (resp. $\mathbf{DA}_{n+m,c}(T)$). In particular, if f is quasi-finite, then $f_!$ preserves the subcategories $\mathbf{DA}_n(-)$ and $\mathbf{DA}_{n,c}(-)$.*

Proof. To treat the case of subcategories of compact objects, we combine the following arguments with the Lemma 3.1.8 and the "easy" constructibility results of [Ayo14a, Proposition 8.5]. Consequently, we only treat the case of $\mathbf{DA}_n(-)$.

Statement (i) follows from the fact that f^*, being a left adjoint, commutes with small sums, from the $\mathrm{Ex}^*_\sharp$ isomorphism and the fact that being of relative dimension $\leq n$ is stable by base change.

The proof of (ii) is the same as that of Proposition 3.1.14 (iv), keeping track of the relative dimensions involved. □

We list some useful corollaries of the results above.

Corollary 3.1.19. *Let $\mathcal{T}(-)$ be one of $\mathbf{DA}^{\mathrm{coh}}(-)$, $\mathbf{DA}_{\mathrm{hom}}(-)$, $\mathbf{DA}^n(-)$, $\mathbf{DA}_n(-)$ or one of their subcategories of compact objects.*

(i) *The system $\mathcal{T}(-)$ localises in the following sense: for $M \in \mathbf{DA}(S)$, $i : Z \to S$ and $j : U \to S$ are complementary closed and open immersions, $M \in \mathcal{T}(S)$ if and only if $i^*M \in \mathcal{T}(Z)$ and $j^*M \in \mathcal{T}(U)$.*

(ii) *Let $f : T \to S$ be a finite radicial surjective morphism (e.g. a nil-immersion), $M \in \mathbf{DA}(S)$, $N \in \mathbf{DA}(T)$. Then $M \in \mathcal{T}(S)$ if and only if $f^*M \in \mathcal{T}(T)$, and $N \in \mathcal{T}(T)$ if and only if $f_*N \in \mathcal{T}(S)$.*

Proof. Statement (i) follows directly from localisation and the permanence properties above. Similarly, statement (ii) follows directly [Ayo07a, Proposition 2.1.163] (which applies because $\mathbf{DA}(-)$ is separated) and the permanence properties. □

Finally, let us discuss what happens with internal Homs and duality.

Corollary 3.1.20. *We have $\underline{\mathsf{Hom}}(\mathbf{DA}_{\mathrm{hom},c}(S), \mathbf{DA}^{\mathrm{coh}}_{(c)}(S)) \subset \mathbf{DA}^{\mathrm{coh}}_{(c)}(S)$. In particular, if S is regular and we take $\mathbb{Q}_S$ as dualizing object, then Verdier duality $\mathbb{D}_S := \underline{\mathsf{Hom}}(-, \mathbb{Q}_S)$ sends compact homological motives to compact cohomological motives.*

Proof. If $M \in \mathbf{DA}(S)$ is compact, then $\underline{\mathsf{Hom}}(M, -)$ commutes with small sums. This shows that we can restrict to generators of $\mathbf{DA}^{\mathrm{coh}}(S)$ in the second variable. Using [Nee01, Lemma 4.4.5], we see that we can restrict to generators of $\mathbf{DA}_{\mathrm{hom},c}(S)$ in the first variable. The result then follows from [Ayo07a, Proposition 2.3.51-52], the $\mathrm{Ex}^*_\sharp$ isomorphism and Proposition 3.1.11 (ii). □

Remark 3.1.21. Even on a regular scheme, the categories of constructible homological and cohomological motives are not anti-equivalent through Verdier duality with dualizing object $\mathbb{Q}$ (see, however, Proposition 3.1.26 below). Indeed, assume S regular of dimension $d > 0$, let $i : x \to S$ be the inclusion of a closed point x and $j : U \to S$ be the complementary open immersion. Then by colocalisation and absolute purity, $j_*\mathbb{Q}_U \in \mathbf{DA}^{\mathrm{coh}}(S)$ sits in a triangle

$$i_*\mathbb{Q}(-d)[-2d] \to \mathbb{Q}_S \to j_*\mathbb{Q}_U \xrightarrow{+} .$$

In particular, it is cohomological. On the other hand, we have $\mathbb{D}_S(\mathbb{Q}_S) \simeq \mathbb{Q}_S \in \mathbf{DA}^{\mathrm{coh}}(S)$ and $\mathbb{D}_S(\, i_!i^!\mathbb{Q}_S) \simeq i_*\mathbb{Q}_S \in \mathbf{DA}^{coh}(S)$, so that by taking the Verdier dual of the triangle above, we have $\mathbb{D}_S(j_*\mathbb{Q}_U) \in \mathbf{DA}^{\mathrm{coh}}(S)$.

If Verdier duality did exchange homological and cohomological motives, we would have $j_*\mathbb{Q}_U \in \mathbf{DA}_{\mathrm{hom}}(S) \cap \mathbf{DA}^{\mathrm{coh}}(S)$ which is equal to $\mathbf{DA}_0(S)$ by Corollary 3.3.7 (ii) below. We would then also have $i_*\mathbb{Q}(-d)[-2d] \in \mathbf{DA}_0(S)$; hence, $i^*i_*\mathbb{Q}(-d) \simeq \mathbb{Q}(-d) \in \mathbf{DA}_0(x)$. This is not the case, as can be seen in a number of ways; for instance, in the proof of Corollary 3.3.7 (iv) we will show that for all $M \in \mathbf{DA}_0(x)$, we have $\mathrm{Hom}(M, \mathbb{Q}(-d)) = 0$.

3.1.3 Continuity

We have a continuity result for subcategories of compact objects.

Proposition 3.1.22. *Let I be a cofiltering small category and $(X_i)_{i\in I} \in \mathsf{Sch}^I$ with affine transition morphisms. Let $X = \varprojlim_{i\in I} X_i$ (X is still assumed to be noetherian and finite-dimensional). Then $\mathbf{DA}_c^{\mathrm{coh}}(X)$ (resp. $\mathbf{DA}_{\mathrm{hom},c}(X)$, $\mathbf{DA}_c^n(X)$, $\mathbf{DA}_{n,c}(X)$, $\mathbf{DA}_{n,c}^{\mathrm{eff}}(X)$) is equal to the $2-$colimit of the $\mathbf{DA}_c^{\mathrm{coh}}(X_i)$ (resp. $\mathbf{DA}_{\mathrm{hom},c}(X_i)$, $\mathbf{DA}_c^n(X_i)$, $\mathbf{DA}_{n,c}(X_i)$, $\mathbf{DA}_{n,c}^{\mathrm{eff}}(X_i)$) via the pullback functors $(X \to X_i)^*$.*

Proof. Using the continuity result for morphisms in **DA** from [Ayo14a, Proposition 3.19] and the arguments from [Ayo, Corollaire 1.A.3] (using the description of compact objects discussed in Lemma 3.1.8), it is enough to prove the following lemma (which extends [Ayo, Lemme 1.A.2]).

Lemma 3.1.23. *With the notation of the proposition, let Y be an X-scheme of finite presentation. Then there exists an $i \in I$ and an X_i-scheme Y_i of finite presentation such that $Y \simeq Y_i \times_{X_i} X$. Moreover, if Y/X is smooth (resp. of relative dimension $\leq n$, smooth of relative dimension $\leq n$), then Y_i can be chosen smooth (resp. of relative dimension $\leq n$, smooth of relative dimension $\leq n$).*

Proof. The first part is well known (see [Gro66b, §8]). For the second part, the arguments of the proof of [Ayo, Lemme 1.A.2] cover the case of smooth and smooth of relative dimension $\leq n$. See [Sta, Tag 05M5] for the case of relative dimension $\leq n$.

□

We deduce a useful punctual characterization of compact n-motives:

Proposition 3.1.24. *Let S be a scheme and $M \in \mathbf{DA}_c(S)$. Then the following are equivalent.*

(i) *$M \in \mathbf{DA}_c^{\mathrm{coh}}(S)$ (resp. $\mathbf{DA}_{\mathrm{hom},c}(S)$, $\mathbf{DA}_c^n(S)$, $\mathbf{DA}_{n,c}(S)$).*

(ii) *For all $s \in S$, we have $s^*M \in \mathbf{DA}_c^{\mathrm{coh}}(s)$ (resp. $\mathbf{DA}_{\mathrm{hom},c}(s)$, $\mathbf{DA}_c^n(s)$, $\mathbf{DA}_{n,c}(s)$).*

Proof. The direction (i)$\Leftarrow$ (ii) follows from the stability established above of all these subcategories by pullbacks. For the other direction, we can assume S reduced by Corollary 3.1.19 (ii). We then proceed by noetherian induction. The case of generic points is settled by the hypothesis, we then use Proposition 3.1.22 to spread-out the property to an open set. We conclude by using Corollary 3.1.19 (i) and the induction hypothesis. □

3.1.4 Over a field

Over a field, Verdier duality does interact well with our subcategories of **DA**.

Lemma 3.1.25. *Let k be a field. Write $\mathbb{D}_k := \underline{\mathsf{Hom}}(-, \mathbb{Q}_k) : \mathbf{DA}(k)^{\mathrm{op}} \to \mathbf{DA}(k)$ for the Verdier duality functor. We have*

$$\mathbb{D}_k(\mathbf{DA}_{\mathrm{hom},c}(k)) \subset \mathbf{DA}_c^{\mathrm{coh}}(k)$$

and $\mathbb{D}_k$ restricts to anti-equivalences of categories

$$\mathbb{D}_k : \mathbf{DA}_{\mathrm{hom},c}^{\mathrm{gsm}}(k) \xrightarrow{\sim} \mathbf{DA}_{\mathrm{gsm},c}^{\mathrm{coh}}(k) \text{ and}$$

$$\mathbb{D}_k : \mathbf{DA}_{n,c}^{\mathrm{gsm}}(k) \xrightarrow{\sim} \mathbf{DA}_{\mathrm{gsm},c}^{n}(k).$$

Proof. For X a separated scheme of finite type over k, consider the more general Verdier duality functor $\mathbb{D}_{X/k} := \underline{\mathsf{Hom}}(-, \pi_X^!\mathbb{Q}_k) : \mathbf{DA}(X)^{\mathrm{op}} \to \mathbf{DA}(X)$. By [Ayo14a, Théorèmes 8.12-8.14], this functor preserves compact objects and its restriction to $\mathbf{DA}_c(X)$ is an anti-autoequivalence which is its own quasi-inverse.

The behaviour of $\mathbb{D}_{X/k}$ with respect to the four operations is explained in [Ayo07a, Théorème 2.3.75]: informally, Verdier duality exchanges f_* and $f_!$, and f^* and $f^!$. Moreover, recall that, for f smooth, relative purity provides an isomorphism $f_\sharp f^* \simeq f_! f^!$. This allows to compute the action of $\mathbb{D}_{X/k}$ on generating families. For instance, we have, for any f smooth, $\mathbb{D}_k(f_\sharp f^*\mathbb{Q}_X) \simeq$

$\mathbb{D}_k(f_! f^! \mathbb{Q}_k) \simeq f_* f^* \mathbb{D}_k(\mathbb{Q}_k) \simeq f_* f^* \mathbb{Q}_k$. which is in $\mathbf{DA}^{\mathrm{coh}}(k)$ by Proposition 3.1.11 (ii). This proves the first inclusion. For the equalities for geometrically smooth subcategories, note that if f is smooth projective (resp. smooth projective of relative dimension $\leq n$), the same computation shows that $\mathbb{D}_k(f_\sharp f^* \mathbb{Q}_X)$ is in $\mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm}}(k)$ (resp. $\mathbf{DA}^n_{\mathrm{gsm}}(k)$). This proves one inclusion of the equalities, and the other follows by involutivity of $\mathbb{D}$. □

As a consequence, when the base is the spectrum of a field, several of the notions we have introduced coincide.

Proposition 3.1.26. *Let k be any field; we have the following equalities.*

$$\mathbf{DA}_{\mathrm{hom}}(k) = \mathbf{DA}^{\mathrm{sm}}_{\mathrm{hom}}(k) = \mathbf{DA}^{\mathrm{gsm}}_{\mathrm{hom}}(k).$$

$$\mathbf{DA}^{\mathrm{coh}}(k) = \mathbf{DA}^{\mathrm{coh}}_{\mathrm{sm}}(k) = \mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm}}(k).$$

$$\mathbf{DA}_n(k) = \mathbf{DA}^{\mathrm{sm}}_n(k) = \mathbf{DA}^{\mathrm{gsm}}_n(k).$$

$$\mathbf{DA}^n(k) = \mathbf{DA}^n_{\mathrm{sm}}(k) = \mathbf{DA}^n_{\mathrm{gsm}}(k).$$

The same equalities hold for the subcategories of compact objects, and $\mathbb{D}_k$ restricts to anti-equivalences of categories:

$$\mathbb{D}_k : \mathbf{DA}_{\mathrm{hom},c}(k) \longleftrightarrow \mathbf{DA}^{\mathrm{coh}}_c(k) : \mathbb{D}_k$$

$$\mathbb{D}_k : \mathbf{DA}_{n,c}(k) \longleftrightarrow \mathbf{DA}^n_c(k) : \mathbb{D}_k$$

Proof. The Verdier duality statement is just a restatement of Lemma 3.1.25 in the light of these equalities.

In each case, we prove equality by proving that the generating family on each side lies in the other. The generating families used in the definitions of these categories are formed of compact objects, hence it suffices to prove the equalities for the subcategories of compact objects. By Lemma 3.1.5, we need only prove the inclusions

$$\mathbf{DA}_{\mathrm{hom},c}(k) \subset \mathbf{DA}^{\mathrm{gsm}}_{\mathrm{hom},c}(k),$$

$$\mathbf{DA}^{\mathrm{coh}}_c(k) \subset \mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm},c}(k),$$

$$\mathbf{DA}_{n,c}(k) \subset \mathbf{DA}^{\mathrm{gsm}}_{n,c}(k) \text{ and}$$

$$\mathbf{DA}^n_c(k) \subset \mathbf{DA}^n_{\mathrm{gsm},c}(k).$$

The key is to prove the following claim: for all $n \in \mathbb{N}$, we have $\mathbb{D}_k(\mathbf{DA}^n_c(k)) \subset \mathbf{DA}^{\mathrm{gsm}}_{n,c}(k)$.

Indeed, assume this claim for the next three paragraphs. Then by looking at generators we also get $\mathbb{D}_k(\mathbf{DA}^{\mathrm{coh}}_c(k)) \subset \mathbf{DA}^{\mathrm{gsm}}_{\mathrm{hom},c}(k)$. By applying $\mathbb{D}_k$ again and the equivalence of categories of Lemma 3.1.25, we get inclusions $\mathbf{DA}^n_c(k) \subset \mathbf{DA}^n_{\mathrm{gsm},c}(k)$ and $\mathbf{DA}^{\mathrm{coh}}_c(k) \subset \mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm},c}(k)$. By applying $\mathbb{D}_k$ to the inclusion $\mathbb{D}_k(\mathbf{DA}_{\mathrm{hom},c}(k)) \subset \mathbf{DA}^{\mathrm{coh}}_c(k)$ of Lemma 3.1.25, we also obtain $\mathbf{DA}_{\mathrm{hom},c}(k) \subset \mathbf{DA}^{\mathrm{hom},c}_{\mathrm{gsm}}(k)$. It remains to see that $\mathbf{DA}^n_c(k) \subset \mathbf{DA}^n_{\mathrm{gsm},c}(k)$, which is slightly less clear.

Let $f : X \to k$ smooth of relative dimension $i \leq n$ (we can reduce to this case by considering connected components of X). By relative purity, we have $f_\sharp \mathbb{Q}_X(-n) \simeq f_! \mathbb{Q}_X(i-n)[2i]$ which is in $\mathbf{DA}^n_c(k)$ by Proposition 3.1.17 and 3.1.10.. This shows that $\mathbf{DA}_{n,c}(k)(-n) \subset \mathbf{DA}^n_c(k) = \mathbf{DA}^n_{\mathrm{gsm},c}(k)$ (the last equality having just been established in the previous paragraph). Applying Verdier duality, we get $\mathbb{D}_k(\mathbf{DA}_{n,c}(k))(n) \subset \mathbb{D}_k(\mathbf{DA}^n_{\mathrm{gsm},c}(k)) = \mathbf{DA}^{\mathrm{gsm}}_{n,c}(k)$.

Another application of relative purity shows that $\mathbf{DA}^{\mathrm{gsm}}_{n,c}(k)(-n) = \mathbf{DA}^n_{\mathrm{gsm},c}(k)$. Putting everything together, we have $\mathbb{D}_k(\mathbf{DA}_{n,c}(k)) \subset \mathbf{DA}^n_{\mathrm{gsm},c}(k) = \mathbb{D}_k(\mathbf{DA}^{\mathrm{gsm}}_{n,c}(k))$ so by involutivity of $\mathbb{D}$ we get the missing inclusion $\mathbf{DA}^n_c(k) \subset \mathbf{DA}^n_{\mathrm{gsm},c}(k)$. This finishes the proof of the proposition modulo the claim.

For simplicity, in the rest of the proof, we write $\pi_Y : Y \to k$ for the structural morphism of any k-scheme Y. Using the generating families, we reformulate the claim as follows: for $\pi_X : X \to k$ proper of relative dimension $\leq n$, we have $\mathbb{D}_k(\pi_{X*}\mathbb{Q}_X) \simeq \pi_{X!}\pi^!_X \mathbb{Q}_k$ in $\mathbf{DA}^{\mathrm{gsm}}_{n,c}(k)$. Let $i : X_{\mathrm{red}} \to X$. Then by localisation we have $\pi_{X!}\pi^!_X \mathbb{Q}_k \simeq \pi_{X!} i_! i^! \pi^!_X \mathbb{Q}_k \simeq \pi_{X_{\mathrm{red}}}!\pi^!_{X_{\mathrm{red}}} \mathbb{Q}_k$. Consequently, we can assume that X is reduced.

We first treat the case of a perfect field k. We proceed by induction on the dimension of X. When X is 0-dimensional, we see that π_X is finite étale because k is perfect and X is reduced, so that $\pi_{X!}\pi_X^! \simeq \pi_{X\sharp}\pi_X^*$ and we are done. For the induction step, we apply De Jong's resolution of singularities by alterations [dJ96, Theorem 4.1 + following remark]. We obtain an alteration $h : \widetilde{X} \to X$ with $\widetilde{X}/k$ a smooth projectivevariety. Recall that h is proper surjective and generically finite. We choose a diagram of schemes with cartesian squares

$$\begin{tikzcd} V \arrow[r, "\tilde{\jmath}"'] \arrow[d, "h_U"] & \widetilde{X} \arrow[d, "h"] & Z \arrow[l, "\tilde{\imath}"] \arrow[d, "h_T"] \\ U \arrow[r, "j"'] & X & T \arrow[l, "i"] \end{tikzcd}$$

with the following properties.

- T is a nowhere dense closed subset of X and U is its open complement.
- h_U can be written as the composite of a purely inseparable finite morphism followed by a finite étale morphism.

Starting from the distinguished colocalisation triangle for the pair (X, U) and applying $\pi_{X!}$, we obtain a triangle

$$\pi_{X!}i_*i^!\pi_X^!\mathbb{Q}_k \to \pi_{X!}\pi_X^!\mathbb{Q}_k \to (\pi_X)_!j_*j^!\pi_X^!\mathbb{Q}_k \xrightarrow{+}$$

that we rewrite as

$$(\pi_T)_!\pi_T^!\mathbb{Q}_k \to \pi_{X!}\pi_X^!\mathbb{Q}_k \to \pi_{X!}j_*\pi_U^!\mathbb{Q}_k \xrightarrow{+} .$$

The left-hand term is in $\mathbf{DA}^{\mathrm{gsm}}_{n,c}(k)$ by induction. To prove that the middle term is in $\mathbf{DA}^{\mathrm{gsm}}_{n,c}(k)$, it remains to prove the same for the right-hand term. Since h_U is finite and the composite of a purely inseparable morphism followed by an étale morphism, the separation property of $\mathbf{DA}$ [Ayo14a, Theorem 3.9] together with [Ayo07a, Corollaire 2.1.164] implies that there is a natural isomorphism of functors:

$$(h_U)_!h_U^! \simeq (h_U)_*h_U^*$$

Now, [Ayo07a, Lemma 2.1.165] implies that $\pi_U^!\mathbb{Q}_k$ is a direct factor of $(h_U)_*h_U^*\pi_X^!\mathbb{Q}_k$. Applying the isomorphism just above, we conclude that $\pi_U^!\mathbb{Q}_k$ is a direct factor of $(h_U)_!h_U^!\pi_U^!\mathbb{Q}_k$. This last motive is isomorphic to $(h_U)_*\pi_V^!\mathbb{Q}_k \simeq (h_U)_*\tilde{\jmath}^*\pi_{\widetilde{X}}^!\mathbb{Q}_k$ because h_U is proper and $\tilde{\jmath}$ is étale. We get that $\pi_{X!}j_*\pi_U^!\mathbb{Q}_k$ is a direct factor of $\pi_{X!}j_*(h_U)_*\tilde{\jmath}^*\pi_{\widetilde{X}}^!\mathbb{Q}_k \simeq \pi_{\widetilde{X}!}\tilde{\jmath}_*\tilde{\jmath}^*\pi_{\widetilde{X}}^!\mathbb{Q}_k$. Applying localisation to the pair $(\widetilde{X}, V)$, the fact that $\widetilde{X}/k$ is smooth projective and the induction hypothesis for Z shows that this last object is in $\mathbf{DA}^{\mathrm{gsm}}_{n,c}(k)$. This concludes the proof when k is perfect.

We now treat the case of a general field k. By the perfect field case and continuity for $\mathbf{DA}^{\mathrm{gsm}}_{n,c}(-)$ (Proposition 3.1.22) applied to the spectrum of the perfect closure of k, we see that there exists a finite purely inseparable extension l/k with $(l/k)^*\pi_{X!}\pi_X^!\mathbb{Q}_k$ in $\mathbf{DA}^{\mathrm{gsm}}_{n,c}(l)$. By the separation property, we have an isomorphism of functors $\mathrm{id} \simeq (l/k)_*(l/k)^*$, so that it is enough to show Lemma 3.1.27 below, which we have already used in the proof of Proposition 3.1.14. □

Lemma 3.1.27. *For a finite field extension l/k and $g : Y \to \mathbf{Spec}(l)$ a smooth projective morphism of relative dimension $\leq n$, there exists a smooth projective variety $g' : Y' \to k$ of dimension $\leq n$ such that $(l/k)_*g_\sharp\mathbb{Q}_Y \simeq g'_\sharp\mathbb{Q}_{Y'} \in \mathbf{DA}^{\mathrm{gsm}}_{n,c}(k)$.*

Proof. We immediately reduce to the case of l/k purely inseparable. Let $F : \mathbf{Spec}(l) \to \mathbf{Spec}(l)$ be an high enough power of the Frobenius of l that factors through k. We denote again by F the induced morphism $\mathbf{Spec}(k) \to \mathbf{Spec}(l)$ and its natural lift $\mathbf{Spec}(k) \to \mathbf{Spec}(k)$ (the corresponding power of Fr_k). We have the following diagram of schemes, where the upper square is cartesian:

$$\begin{tikzcd}
Y' \arrow[r, "F_Y"] \arrow[d, "\pi_{Y'}"'] & Y \arrow[d, "\pi_Y"] \\
\mathbf{Spec}(k) \arrow[r, "F"] \arrow[dr, "F"'] & \mathbf{Spec}(l) \arrow[d, "(l/k)"] \\
& \mathbf{Spec}(k).
\end{tikzcd}$$

By base change, the k-scheme Y' is smooth projective and the morphism F_Y is finite purely inseparable. By the separation property of $\mathbf{DA}$, we have

$$(l/k)_*(\pi_Y)_*\mathbb{Q}_Y \simeq (l/k)_*(\pi_Y)_*(F_Y)_*\mathbb{Q}_{Y'} \simeq (l/k)_*F_*(\pi_{Y'})_*\mathbb{Q}_{Y'} \simeq F_*(\pi_{Y'})_*\mathbb{Q}_{Y'}.$$

Let $\mathrm{Fr}_{Y'}$ be the corresponding power of the absolute Frobenius on Y'. By naturality of the absolute Frobenius, we have $\pi_{Y'} \circ \mathrm{Fr}_{Y'} = F \circ \pi_{Y'} : Y' \to \mathbf{Spec}(k)$. We deduce that

$$F_*(\pi_{Y'})_*\mathbb{Q}_{Y'} \simeq (\pi_{Y'})_*(\mathrm{Fr}_{Y'})_*\mathbb{Q}_{Y'} \simeq (\pi_{Y'})_*\mathbb{Q}_{Y'} \in \mathbf{DA}^n_{\mathrm{gsm}}(k),$$

where the last isomorphism follows by separation. By relative purity and the projection formula, we deduce that

$$\begin{aligned}
(l/k)_*(\pi_Y)_\sharp\mathbb{Q}_Y &\simeq (l/k)_*((\pi_Y)_*\mathbb{Q}_Y \otimes \mathbb{Q}_l(1)[2]) \\
&\simeq (l/k)_*((\pi_Y)_*\mathbb{Q}_Y) \otimes \mathbb{Q}_k(1)[2] \\
&\simeq (\pi_{Y'})_*\mathbb{Q}_{Y'} \otimes \mathbb{Q}_k(1)[2] \\
&\simeq (\pi_{Y'})_\sharp\mathbb{Q}_{Y'}.
\end{aligned}$$

This completes the proof of the lemma. □

3.1.5 Homological vs cohomological motives

Proposition 3.1.28. *Let S be a scheme, $n \geq 0$. We have*

$$\mathbf{DA}^n(S) = \mathbf{DA}_n(S)(-n)$$

and

$$\mathbf{DA}^n_c(S) = \mathbf{DA}_{n,c}(S)(-n).$$

Proof. In both directions, it is enough to check the inclusion for a family of compact generators.

Let $f : X \to S$ be a smooth morphism of relative dimension $i \leq n$ (we can reduce to this case by considering connected components of S and X). By relative purity, we have

$$f_\sharp\mathbb{Q}_X(-n) \simeq f_!\mathbb{Q}_X(i-n)[2i]$$

which is in $\mathbf{DA}^n(S)$ by Proposition 3.1.17 and 3.1.10.

The other inclusion is true for smooth cohomological n-motives by the same relative purity argument. For general compact cohomological n-motives (which include the generating family), we argue as follows. By Corollary 3.1.19 (ii), we can assume S reduced. We then proceed by noetherian induction. Let $M \in \mathbf{DA}^n(S)$. The restriction of M to any generic point of S is smooth by Proposition 3.1.26. There we can apply the smooth case and see that $\eta^*M \in \mathbf{DA}_{n,c}(\eta)(-n)$ for any generic point η of S. Then we apply continuity for compact homological n-motives (Proposition 3.1.22) to find a dense open immersion $j : U \to S$ with $j^*M \in \mathbf{DA}_{n,c}(U)(-n)$. Applying the induction hypothesis, localisation and the fact that i_* preserves homological n-motives for i closed immersion (Proposition 3.1.18 (ii)) completes the proof. □

3.1.6 Nearby cycles

To conclude this section, we prove a result about the nearby cycles functor and n-motives.

Proposition 3.1.29. *Let R be an excellent henselian discrete valuation ring and let $S = \mathbf{Spec}(R)$, η be the generic fiber and σ be the closed fiber. Fix a separable closure K^{sep} of $K = \mathrm{Frac}(R)$ and let $\bar{\sigma}$ be the spectrum of its residue field . Associated to the choice of a uniformizer π of R, there is a tame nearby motive functor $\Psi_\pi^{\mathrm{mod}} : \mathbf{DA}(\eta) \to \mathbf{DA}(\sigma)$ and a nearby motive functor $\Psi_\pi : \mathbf{DA}(\eta) \to \mathbf{DA}(\bar{\sigma})$ (see [Ayo14a, Section 10, Définition 10.14]). Then the functor Ψ_π^{mod} (resp. Ψ_π) :*

1. *sends $\mathbf{DA}^{\mathrm{coh}}(\eta)$ to $\mathbf{DA}^{\mathrm{coh}}(\sigma)$ (resp. to $\mathbf{DA}^{\mathrm{coh}}(\bar{\sigma})$),*
2. *sends $\mathbf{DA}^{n}(\eta)$ to $\mathbf{DA}^{n}(\sigma)$ (resp. to $\mathbf{DA}^{n}(\bar{\sigma})$) for any $n \geq 0$,*
3. *sends $\mathbf{DA}_{\mathrm{hom}}(\eta)$ to $\mathbf{DA}_{\mathrm{hom}}(\sigma)$ (resp. to $\mathbf{DA}_{\mathrm{hom}}(\bar{\sigma})$), and*
4. *sends $\mathbf{DA}_{n}(\eta)$ to $\mathbf{DA}_{n}(\sigma)$ (resp. to $\mathbf{DA}_{n}(\bar{\sigma})$).*

Similar results hold for the subcategories of compact objects.

Proof. The results for compact objects follow from our description of compact objects in these subcategories together with the constructibility theorems for Ψ_f^{mod} proved in [Ayo14a, §10].

We need to work with a more general set-up. Let X be an S-scheme and $f : X \to \mathbb{A}^1_S$ a morphism. Recall that the functor Ψ_f of [Ayo14a, Définition 10.14] is constructed by a two step process: first a functor of "tame nearby cycles" $\Psi_f^{\mathrm{mod}} : \mathbf{DA}(X_\eta) \to \mathbf{DA}(X_\sigma)$ is constructed as a special case of [Ayo07b, Définition 3.2.3] and then Ψ_f is obtained via an homotopy colimit along all the finite extensions of K contained in a fixed maximum p-primary extension inside K^{sep} of the maximal unramified extension K^{nr} (such extensions exist by the theorem of Schur-Zassenhaus).

We make this second step slightly more explicit. If M_δ/K^{nr} is such a fixed maximum p-primary extension, let $\mathcal{L}$ be the poset of all the finite subextensions $K \subset L \subset M_\delta$ ordered by the reverse of inclusion. Then there is a diagram of schemes $(T_L, \mathcal{L})$ where T_L is the normalisation of S inside L/K, along with diagrams $(\eta_L, \mathcal{L})$ and $(\sigma_L, \mathcal{L})$ of generic and special fibers. We have a morphism $\gamma : (T_L, \mathcal{L}) \to S$ and we pullback the diagram of schemes over S used to compute Ψ_f^{mod} along this morphism. We also use the notation $\widetilde{\mathcal{L}} = \mathcal{L} \times \Delta \times (\mathbb{N}')^\times$ where Δ is the simplicial category and $(\mathbb{N}')^\times = \{n \in \mathbb{N}^\times | car(\sigma) \nmid n\}$. Altogether, we get a commutative diagram of diagrams of schemes with cartesian squares:

$$\begin{tikzcd}
(\mathcal{R}'_{f_L}, \widetilde{\mathcal{L}}) \arrow[r, "\theta^{\mathcal{R}'}_{f_{\mathcal{L}}}"] \arrow[d, "f_{\eta_{\mathcal{L}}}"] & (X_{\eta_L}, \widetilde{\mathcal{L}}) \arrow[r, "j"] \arrow[d, "f_{\eta_{\mathcal{L}}}"] & (X_{T_L}, \widetilde{\mathcal{L}}) \arrow[d, "f_{\mathcal{L}}"] & (X_{\sigma_L}, \widetilde{\mathcal{L}}) \arrow[l, "i"'] \arrow[r, "p_{\Delta\times(\mathbb{N}')^\times}"] \arrow[d, "f_{\sigma_{\mathcal{L}}}"] & (X_{\sigma_L}, \mathcal{L}) \arrow[d, "f_{\sigma_{\mathcal{L}}}"] \\
(\mathcal{R}'_{T_L}, \widetilde{\mathcal{L}}) \arrow[r, "\theta^{\mathcal{R}'}_{\mathcal{L}}"] & ((\mathbb{G}_{\mathrm{m}})_{T_L}, \widetilde{\mathcal{L}}) \arrow[r, "j"] & (\mathbb{A}^1_{T_L}, \widetilde{\mathcal{L}}) & (T_L, \widetilde{\mathcal{L}}) \arrow[l, "i"'] \arrow[r, "p_{\Delta\times(\mathbb{N}')^\times}"] & (T_L, \mathcal{L})
\end{tikzcd}$$

We also have morphisms $\bar{\pi} : (X_{\bar{\sigma}}, \mathcal{L}) \to (X_{\sigma_L}, \mathcal{L})$, $p_{\mathcal{L}} : (X_{\bar{\sigma}}, \mathcal{L}) \to X_{\bar{\sigma}})$ and $\gamma_{X_\eta} : (X_{\eta_L}, \mathcal{L}) \to X_\eta$. We can finally define:

$$\Psi_f = (p_{\mathcal{L}})_\sharp \bar{\pi}^* (p_{\Delta\times(\mathbb{N}')^\times})_\sharp i^* j_* (\theta^{\mathcal{R}'}_{f_{\mathcal{L}}})_* (\theta^{\mathcal{R}'}_{f_{\mathcal{L}}})^* (p_{\Delta\times(\mathbb{N}')^\times})^* \gamma^*_{X_\eta}$$

This formula and the similar one defining Ψ_f^{mod} [Ayo14a, Formula (97)] imply that both functors commute with small sums. Using the fact that nearby cycles commute with duality on constructible objects [Ayo14a, Theorème 10.20] together with Lemma 3.1.25 allows us to deduce (iii) and (iv) from (i) and (ii) by a duality argument.

It remains to show the property (i) (resp. (ii)) for the compact generators of $\mathbf{DA}^{\mathrm{coh}}(\eta)$ of the form $f_*\mathbb{Q}_X$ with $f : X \to \eta$ proper (resp. of $\mathbf{DA}^{n}(\eta)$ of the form $f_*\mathbb{Q}_X$ with $f : X \to \eta$ a proper morphism of relative dimension $\leq n$). Moreover, since these objects are constructible, [Ayo14a, Théorème 10.13] implies that the conclusion for Ψ_π is implied by the one for Ψ_π^{mod}, so we concentrate on moderate nearby cycles.

Let $f^0 : X^0 \to \eta$ be proper of dimension $\leq n$. We show show that $\Psi_\pi^{\text{mod}} f^0_* \mathbb{Q}_{X^0}$ is in $\mathbf{DA}^n(\sigma)$ by induction on n: this is enough to prove (i) and (ii). Choose a proper flat morphism $f : X \to S$ such that $X^0 = X_\eta$ and $f^0 = f_\eta$. Using normalisation, localisation and the induction hypothesis, we reduce to the case where X_η (hence X) is irreducible.

The special fiber X_σ is also of relative dimension $\leq n$. Because Ψ^{mod} is a specialisation system [Ayo07b, Definition 3.1.1] and f is proper, we have an isomorphism

$$\beta : \Psi_\pi^{\text{mod}} (f_\eta)_* \mathbb{Q}_{X_\eta} \xrightarrow{\sim} (f_\sigma)_* \Psi_f^{\text{mod}} \mathbb{Q}_{X_\eta}.$$

To simplify the notation, for any S-scheme $g : W \to S$, we write

$$\Psi_W := (g_\sigma)_* \Psi_f^{\text{mod}} \mathbb{Q}_{X_\eta}.$$

Now we want to reduce to a situation with a better behaved special fiber. We apply De Jong's theorem on semi-stable reduction by alterations [dJ96, Theorem 6.5]. There exists an henselian DVR $\tilde{S}$ finite over S and a commutative square

$$\begin{tikzcd} \tilde{X} \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \tilde{S} \arrow[r, "\pi"'] & S \end{tikzcd}$$

such that

- $\tilde{X}$ is regular and strictly semi-stable over $\tilde{S}$ (in the sense of [dJ96, 2.16]), and
- p is an alteration.

Let V be an open set contained in $\tilde{X}_\eta$ such that p_V is the composition of a finite flat purely inseparable morphism followed by a finite étale morphism, and consider $Z = \tilde{X} \setminus V$ with its reduced scheme structure. We have a commutative diagram (not necessarily cartesian, but with $U := p(V)$ open by flatness and $T := p(Z) = X \setminus U$ by surjectivity)

$$\begin{tikzcd} V \arrow[r, "\tilde{j}"] \arrow[d, "p_U"'] & \tilde{X} \arrow[d, "p"] & Z \arrow[l, "\tilde{i}"'] \arrow[d, "p_Z"] \\ U \arrow[r, "j"'] & X & T \arrow[l, "i"] . \end{tikzcd}$$

We have distinguished triangles in $\mathbf{DA}(X_\eta)$,

$$(j_\eta)_! \mathbb{Q}_{U_\eta} \to \mathbb{Q}_{X_\eta} \to (i_\eta)_* \mathbb{Q}_{T_\eta} \xrightarrow{+}$$

and

$$(p_{U_\eta})_* (\tilde{j}_\eta)_! \mathbb{Q}_{V_\eta} \to (p_\eta)_* \mathbb{Q}_{X'_\eta} \to (i_\eta)_* (p_{Z_\eta})_* \mathbb{Q}_{Z_\eta} \xrightarrow{+} .$$

After applying Ψ_f^{mod}, pushing forward to σ (we forget temporarily the $\tilde{S}$-scheme structure of X', V and Z) and using properness of p and i, we get distinguished triangles

$$(f_\sigma)_* \Psi_f^{\text{mod}} (j_\eta)_! \mathbb{Q}_{U_\eta}) \to \Psi_X \to \Psi_T \xrightarrow{+}$$

and

$$(f_\sigma)_* \Psi_f^{\text{mod}} (p_{U_\eta})_* (\tilde{j}_\eta)_! V_\eta \to \Psi_{\tilde{X}} \to \Psi_Z \xrightarrow{+}$$

in $\mathbf{DA}(\sigma)$.

Since $\dim(T_\eta), \dim(Z_\eta) < n$, the terms involving Ψ_T and Ψ_Z are handled by induction. On the other hand, we know that by separation, [Ayo07a, Lemme 2.1.165], $(j_\eta)_! \mathbb{Q}_{U_\eta}$ is a direct factor of $(p_{U_\eta})_* (\tilde{j}_\eta)_! V_\eta$. It is thus enough to prove the result for $\Psi_{\tilde{X}}$.

We will prove that $(g_\sigma)_* \Psi_g^{\text{mod}} \mathbb{Q}_{X'_\eta} \in \mathbf{DA}^n(\tilde{S})$. Since $\pi : \tilde{S} \to S$ is finite, this will imply the same for $\Psi_{\tilde{X}}$ by Proposition 3.1.17 (ii).

Write $\tilde{X}_s = \cup_{k=1}^m D_k$ as a union of its irreducible components. For each $I \subset [1,m]$ write $D_I = \bigcap_{k\in I} D_k$ for the scheme-theoretic intersection. For $I \subset J$, write $(i_J^I) : D_J \to D_I$ for the corresponding closed immersion. For any I, write

$$D_I^\circ := D_I \setminus \bigcup_{l\notin I} D_l$$

and $j_I : D_I^\circ \to D_I$ for the corresponding open immersion.

By [Ayo07a, Lemme 2.2.31], it is enough to prove that for any $I \neq \emptyset$, we have

$$(f_\sigma)_*(i_I)_* i_I^! \Psi_g^{\mathrm{mod}} \mathbb{Q}_{\tilde{X}_\eta} \in \mathbf{DA}^n(\sigma).$$

Let I be such an index set, and let $k \in I$. By relative purity for the regular codimension 1 closed immersion $i_k^!$ and [Ayo07b, Theorem 3.3.43] (which applies to $\mathbf{DA}^{\text{ét}}(-,\mathbb{Q})$), we have

$$\begin{aligned} i_I^! \Psi_g \mathbb{Q}_{\tilde{X}_\eta} &\simeq (i_I^k)^! i_k^! \Psi_g \mathbb{Q}_{\tilde{X}_\eta} \\ &\simeq (i_I^k)^* i_k^* \Psi_g \mathbb{Q}_{\tilde{X}_\eta}(1)[2] \\ &\simeq (i_I^k)^* (j_k)_* (j_k)^* i_k^* \Psi_g \mathbb{Q}_{\tilde{X}_\eta}(1)[2]. \end{aligned}$$

Because D_k is smooth over σ by semi-stability, we have $(j_k)^*(i_k)^!\Psi_g\mathbb{Q}_{\tilde{X}_\eta} \simeq (i_k \circ j_k)^!\Psi_g\mathbb{Q}_{\tilde{X}_\eta} \simeq (i_k \circ j_k)^*\Psi_g\mathbb{Q}_{\tilde{X}_\eta} \simeq \mathbb{Q}_{D_k^\circ}$ by axiom (SPE2).2 of specialization systems [Ayo07b, Definition 3.1.1] and [Ayo14a, Theorem 10.6] (with $e = 1$). So we are reduced to computing $(i_I^k)^*(j_k)_*\mathbb{Q}_{D_k^\circ}$. Since everything involved is smooth over σ, relative purity, localisation and a further induction on branches imply that this motive is an iterated extension of sums of negative Tate twists $\mathbb{Q}_{D_I}(-d)[-2d]$ for $d \leq \mathrm{card}(I \setminus \{k\})$, so it is in $\mathbf{DA}^{\mathrm{card}(I)-1}(D_I)$ by Proposition 3.1.10 (iii). Since D_I is of relative dimension $\leq n - \mathrm{card}(I)$ over σ, an application of Proposition 3.1.17 (ii) and the observation that $1 + (\mathrm{card}(I) - 1) + (n - \mathrm{card}(I)) = n$ finishes the proof. □

3.2 Commutative group schemes and motives

Several motives of interest for this paper are obtained from group schemes or complexes of group schemes. The main examples we are interested in are smooth commutative group schemes, Deligne 1-motives (Appendix 3.A), and the smooth Picard complex (Section 3.2.3).

3.2.1 Motives of commutative group schemes

In this section, we introduce the relevant definitions and reformulate results from [AHPL14] and [Org04] in this language. For the rest of the section, fix a noetherian finite-dimensional scheme S.

Definition 3.2.1. The subcategory $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{gr}}(S)$ (resp. $\mathbf{DA}_{\mathrm{gr}}(S)$) of $\mathbf{DA}^{\mathrm{eff}}(S)$ (resp. $\mathbf{DA}(S)$) is the triangulated subcategory generated by motives of the form $G \otimes \mathbb{Q}$ (resp. $\Sigma^\infty(G \otimes \mathbb{Q})$) with G smooth (locally of finite type) commutative group scheme over S.

In [AHPL14, Thm D.1], we constructed a functorial cofibrant resolution of the sheaf $G \otimes \mathbb{Q}$ for G a smooth (locally of finite type) commutative group scheme over S. Let us recall the statement.

Lemma 3.2.2. *[AHPL14, Thm D.1] Let $(\mathcal{S}, \tau)$ be a Grothendieck site. We denote $\mathbb{Z}(-)$ the functor "free abelian group sheaf" (the sheafification of the sectionwise free abelian group functor).*

There is a functor:

$$A : \mathbf{Sh}_\tau(\mathcal{S}, \mathbb{Z}) \to \mathbf{Cpl}_{\geq 0}\mathbf{Sh}_\tau(\mathcal{S}, \mathbb{Z})$$

together with a natural transformation

$$r : A \to (-)[0]$$

satisfying the following properties.

1. *For all $\mathcal{G} \in \mathbf{Sh}_\tau(S,\mathbb{Z})$ and $i \geq 0$, the sheaf $A(\mathcal{G})_i$ is of the form $\bigoplus_{j=0}^{d(i)} \mathbb{Z}(\mathcal{G}^{a(i,j)})$ for some $d(i), a(i,j) \in \mathbb{N}$.*

2. *There is a natural transformation $\tilde{a} : \mathbb{Z}(-)[0] \to A$ which lifts the addition map $a : \mathbb{Z}(-) \to \mathrm{id}$; that is, one has $a[0] = r\tilde{a}$.*

3. *The functor A and the transformations r and $\tilde{a}$ are compatible with pullbacks by morphisms of sites.*

4. *The map $r \otimes \mathbb{Q}$ is a quasi-isomorphism.*

Let us make more explicit the statement in 3. Recall that we use underlines to denote underived functors between categories of complexes. For a morphism of sites $F : \mathcal{S}' \to \mathcal{S}$, and $\mathcal{G}$ as in the theorem, we assert that there exists an isomorphism of complexes $b_{F,\mathcal{G}} : \underline{F}^*(A(\mathcal{G})) \to A(\underline{F}^*(\mathcal{G}))$ which is termwise compatible with the standard isomorphisms $\underline{F}^*\mathbb{Z}(\mathcal{G}^{a(i,j)}) \simeq \mathbb{Z}(\underline{F}^*\mathcal{G}^{a(i,j)})$ and which makes the diagram

$$\begin{tikzcd} \underline{F}^*(A(\mathcal{G})) \arrow[r, "\underline{F}^*(r(\mathcal{G}))"] \arrow[d, "b_{F,\mathcal{G}}"'] & \underline{F}^*\mathcal{G} \\ A(\underline{F}^*\mathcal{G}) \arrow[ur, "r(\underline{F}^*\mathcal{G})"'] & \end{tikzcd}$$

commute.

Corollary 3.2.3. *Let $f : T \to S$ be a morphism of schemes. Then $f^* \mathbf{DA}_{\mathrm{gr}}(S) \subset \mathbf{DA}_{\mathrm{gr}}(T)$.*

Proposition 3.2.4. *Let K_* be a bounded complex of smooth commutative group schemes over S and $f : T \to S$ a morphism of schemes. We have a natural isomorphism*

$$R_f : f^*(K_* \otimes \mathbb{Q}) \xrightarrow{\sim} \underline{f}^*(K_* \otimes \mathbb{Q})$$

in $D(\mathsf{Sm}/S)$. Moreover, R_f is compatible with further pullbacks: for $g : U \to T$, the diagram

$$\begin{tikzcd} g^*f^*(K_* \otimes \mathbb{Q}) \arrow[r, "\sim"'] \arrow[d, "R_f"', "\sim"] & (fg)^*(K_* \otimes \mathbb{Q}) \arrow[r, "R_{fg}", "\sim"'] & \underline{(fg)}^*(K_* \otimes \mathbb{Q}) \arrow[d, "\sim"] \\ g^*\underline{f}^*(K_* \otimes \mathbb{Q}) \arrow[rr, "\sim", "R_g"'] & & \underline{g}^*\underline{f}^*(K_* \otimes \mathbb{Q}) \end{tikzcd}$$

commutes.

Proof. We apply Lemma 3.2.2 to the individual sheaves K_n, and use the natural functoriality of the construction. This yields a double complex $A(K_*)_{i\in\mathbb{Z},j\in\mathbb{N}}$ together with a map $r : A(K_*) \to K_*$. We then tensor by $\mathbb{Q}$ and take the total complex along the second index. This yields a complex $B_\mathbb{Q}(K)_*$ of sheaves of $\mathbb{Q}$-vector spaces on $(\mathsf{Sm}/S)_{\text{ét}}$ together with a map $r_\mathbb{Q}(K_*) : B_\mathbb{Q}(K)_* \to K_*$ with the following properties.

(i) For all $i \in \mathbb{Z}$, the sheaf $B_\mathbb{Q}(K)_i$ is of the form $\mathbb{Q}(H_i)$ for some smooth commutative group scheme H_i over S (a fiber product of various copies of the K_n's); therefore, $B_\mathbb{Q}(K_*)$ is a projective object in $\mathbf{Cpl}(\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}/S,\mathbb{Q}))$.

(ii) The map $r_\mathbb{Q}(K_*)$ is a quasi-isomorphism, hence a projective resolution of $K_* \otimes \mathbb{Q}$.

(iii) The formation of $B_\mathbb{Q}(K_*)$ and $r_\mathbb{Q}(K_*)$ is compatible with (underived) pullback, in the sense that, for any morphism $f : T \to S$, there exists an isomorphism of complexes b_{f,K_*} : $\underline{f}^*(B_\mathbb{Q}(K_*)) \to B_\mathbb{Q}(\underline{f}^*K_*)$ which makes the following diagram in $\mathbf{Cpl}(\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}/T,\mathbb{Q}))$ commutes.

$$\begin{tikzcd} \underline{f}^*(B_\mathbb{Q}(K_*)) \arrow[r, "\underline{f}^*(r_\mathbb{Q}(K_*))"] \arrow[d, "b_{f,K_*}"'] & \underline{f}^*(K_* \otimes \mathbb{Q}) \\ B_\mathbb{Q}(\underline{f}^*(K_*)) \arrow[ur, "r(\underline{f}^*K_*)"'] & \end{tikzcd}$$

Because $r_{\mathbb{Q}}$ is a projective resolution, we have an isomorphism in $D(\mathsf{Sm}/S)$

$$f^*(K_* \otimes \mathbb{Q}) \xleftarrow[\sim]{f^*(r_{\mathbb{Q}}(K_*))} f^*(B_{\mathbb{Q}(K_*)}) \simeq \underline{f}^*(B_{\mathbb{Q}}(K_*)).$$

We define R_f as the composition

$$f^*(K_* \otimes \mathbb{Q}) \xrightarrow[\sim]{f^*(r_{\mathbb{Q}}(K_*))^{-1}} \underline{f}^*(B_{\mathbb{Q}}(K_*)) \xrightarrow[\sim]{\underline{f}^*(r_{\mathbb{Q}})} \underline{f}^*(K_* \otimes \mathbb{Q}).$$

It remains to check the compatibility with further pullbacks. Let $g : U \to T$ be a morphism of schemes. The reader is invited to contemplate the following diagram in $D(\mathsf{Sm}/S)$ (where the unlabelled maps are either cocycle isomorphisms for the pullbacks - derived and not - or isomorphisms of the form $h^*(C) \simeq \underline{h}^*(C)$ for C cofibrant).

$$\begin{tikzcd}
g^*f^*(K_* \otimes \mathbb{Q}) \arrow[d, "\sim"'] \arrow[dr, phantom, "\textbf{(A)}"] & g^*f^*B_{\mathbb{Q}}(K_*) \arrow[l, "g^*f^*r_{\mathbb{Q}}"', "\sim"] \arrow[ddl, "\sim"'] \arrow[dr, phantom, "\textbf{(B)}"] & g^*\underline{f}^*B_{\mathbb{Q}}(K_*) \arrow[l, "\sim"'] \arrow[r, "g^*\underline{f}^*r_{\mathbb{Q}}"] \arrow[dr, phantom, "\textbf{(C)}"] & g^*\underline{f}^*(K_* \otimes \mathbb{Q}) \\
(fg)^*(K_* \otimes \mathbb{Q}) & & & \\
(fg)^*B_{\mathbb{Q}}(K_*) \arrow[u, "(fg)^*r_{\mathbb{Q}}"] \arrow[dr, phantom, "\textbf{(D)}"] & & & g^*B_{\mathbb{Q}}(\underline{f}^*(K_*)) \arrow[uul, "g^*b_{f,K_*}"'] \arrow[uu, "g^*r_{\mathbb{Q}}"'] \arrow[l, phantom, "\textbf{(E)}"] \\
(\underline{f}\underline{g})^*B_{\mathbb{Q}}(K_*) \arrow[u, "\sim"'] \arrow[d, "(\underline{f}\underline{g})^*r_{\mathbb{Q}}"'] \arrow[dr, phantom, "\textbf{(F)}"] & \underline{g}^*\underline{f}^*B_{\mathbb{Q}}(K_*) \arrow[l, "\sim"] \arrow[uuu, "\sim"] \arrow[uuur, "\sim"] \arrow[drr, "\underline{g}^*\underline{f}^*r_{\mathbb{Q}}"'] & & \underline{g}^*B_{\mathbb{Q}}(\underline{f}^*K_*) \arrow[u, "\sim"'] \arrow[ll, "\underline{g}^*b_{f,K_*}"'] \arrow[d, "\underline{g}^*r_{\mathbb{Q}}"] \arrow[dl, phantom, "\textbf{(G)}"] \\
(\underline{f}\underline{g})^*(K_* \otimes \mathbb{Q}) & & & \underline{g}^*\underline{f}^*(K_* \otimes \mathbb{Q}) \arrow[lll, "\sim"]
\end{tikzcd}$$

The quadrangles (A) and (F) commute because of the naturality of the cocycle isomorphisms for pullbacks. The triangle (B) and the quadrangle (E) commute trivially. The triangles (C) and (G) commute because of property (iii) above. Finally, the quadrangle (D) commutes because the cocycle isomorphisms for derived and underived pullbacks are compatible. $\square$

Corollary 3.2.5. *Let K_* be a bounded complex of smooth commutative group schemes over S and $f : T \to S$ be a morphism of schemes. We have natural isomorphisms*

$$R_f : f^*K_* \otimes \mathbb{Q} \xrightarrow{\sim} \underline{f}^*(K_* \otimes \mathbb{Q})$$

in $\mathbf{DA}^{\mathrm{eff}}(S)$ and

$$R_f : f^*\Sigma^\infty(K_* \otimes \mathbb{Q}) \xrightarrow{\sim} \Sigma^\infty \underline{f}^*(K_* \otimes \mathbb{Q})$$

in $\mathbf{DA}(S)$. These isomorphisms are compatible with further pullbacks in the same way as in the previous proposition.

Proof. The first isomorphism follows directly from Proposition 3.2.4. The second follows from the first together with the commutation of f^* and Σ^∞. $\square$

For some arguments, we need to use motives with transfers of commutative group schemes over a field.

Definition 3.2.6. Let k be a field and G a smooth (locally of finite type) commutative group scheme over k. Recall that the étale sheaf G on Sm/S admits a canonical structure of sheaf with transfers [BVK10, Lemma 1.4.4], which is functorial in G. We write G^{tr} for the resulting sheaf with transfers. We then define $\mathbf{DM}^{(\mathrm{eff})}_{\mathrm{gr}}(k) \subset \mathbf{DM}^{(\mathrm{eff})}(k)$ by analogy with Definition 3.2.1.

Recall that there are adjunctions

$$a_{\mathrm{tr}} : \mathbf{DA}^{(\mathrm{eff})}(k) \leftrightarrows \mathbf{DM}^{(\mathrm{eff})}(k) : o^{\mathrm{tr}}$$

which relate motives with and without transfers.

Proposition 3.2.7. *Let $M \in \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gr}}(k)$. Then the counit morphisms*

$$a_{\mathrm{tr}} o^{\mathrm{tr}}(M \otimes \mathbb{Q}) \xrightarrow{\sim} M \otimes \mathbb{Q}$$

in $\mathbf{DM}^{\mathrm{eff}}(k)$ and

$$a_{\mathrm{tr}} o^{\mathrm{tr}} \Sigma^{\infty}_{\mathrm{tr}}(M \otimes \mathbb{Q}) \xrightarrow{\sim} \Sigma^{\infty}_{\mathrm{tr}}(M \otimes \mathbb{Q}) \text{ in } \mathbf{DM}(k)$$

in $\mathbf{DM}(k)$ are isomorphisms.

Proof. We reduce immediately to the case of $M = G^{\mathrm{tr}}_S \otimes \mathbb{Q}$, which is covered by [AHPL14, Proposition 2.10]. □

An important consequence for us is the following computation, which consists of translating a classical result of Voevodsky to our context, and which we will generalize later on.

Proposition 3.2.8. *Let C/k be a smooth projective curve. There exists a direct sum decomposition*

$$M(C) \simeq \mathbb{Q} \oplus \Sigma^{\infty}(\mathrm{Jac}(C)_{\mathbb{Q}}) \oplus \mathbb{Q}(1)[2]$$

in $\mathbf{DA}(k)$.

Proof. We first assume k perfect. For a smooth projective curve C over k with a rational point, Voevodsky has computed the motive $M^{\mathrm{eff}}_{\mathrm{tr}}(C) \in \mathbf{DM}^{\mathrm{eff}}(k)$ (see e.g. [BVK10, Proposition 2.5.5]) and shown that

$$M^{\mathrm{eff}}_{\mathrm{tr}}(C) \simeq \mathbb{Q} \oplus (\mathrm{Jac}(C)^{\mathrm{tr}}_{\mathbb{Q}}) \oplus \mathbb{Q}(1)[2].$$

The same argument works with a 0-cycle of degree 1, which exists because we allow rational coefficients. By Proposition 3.2.7, we have $J(C)^{\mathrm{tr}} \simeq a^{\mathrm{tr}} o^{\mathrm{tr}} J(C)^{\mathrm{tr}} \simeq a^{\mathrm{tr}} \underline{o}^{\mathrm{tr}} J(C)^{\mathrm{tr}} \simeq a^{\mathrm{tr}} J(C)$ (because $\underline{o}^{\mathrm{tr}}$ preserves $\mathbb{A}^1$-equivalences [Ayoara, Lemme 2.111]). Applying $\Sigma^{\infty}_{\mathrm{tr}}$ and using that a^{tr} commutes with suspension, we get

$$M_{\mathrm{tr}}(C) \simeq \mathbb{Q} \oplus a^{\mathrm{tr}} \Sigma^{\infty}(\mathrm{Jac}(C)_{\mathbb{Q}}) \oplus \mathbb{Q}(1)[2]$$

in $\mathbf{DM}(k)$. The adjunction $a^{\mathrm{tr}} : \mathbf{DA}(k) \leftrightarrows \mathbf{DM}(k) : o^{\mathrm{tr}}$ is an equivalence of categories by [CDb, Corollary 16.2.22]. This implies that $o^{\mathrm{tr}} M_{\mathrm{tr}}(C) \simeq o^{\mathrm{tr}} a^{\mathrm{tr}} M(C) \simeq M(C)$ and similarly $o^{\mathrm{tr}}\mathbb{Q} \simeq \mathbb{Q}$ and $o^{\mathrm{tr}}\mathbb{Q}(1)[2] \simeq \mathbb{Q}(1)[2]$. Applying o^{tr} to the isomorphism above, we thus get an isomorphism

$$M(C) \simeq \mathbb{Q} \oplus \Sigma^{\infty}(\mathrm{Jac}(C)_{\mathbb{Q}}) \oplus \mathbb{Q}(1)[2]$$

as required.

For a general k, the result follows from the perfect case by separation, continuity for $\mathbf{DA}(-)$ and Proposition 3.2.4. □

We also need an alternative description of the motive $\Sigma^{\infty}(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q})$ which is a relative, rational version of the standard description of the motivic complex $\mathbb{Z}(1)$.

Proposition 3.2.9. *There is a canonical isomorphism*

$$u_S : \Sigma^{\infty}(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}_S(1)[1]$$

in $\mathbf{DA}(S)$. The isomorphism u_S is compatible with pullbacks and the isomorphisms R_f of Corollary 3.2.2: for $f : T \to S$, the diagram

$$\begin{tikzcd}
f^*\Sigma^{\infty}(\mathbb{G}_{m,S} \otimes \mathbb{Q}) \arrow[r, "R_f", "\sim"'] \arrow[d, "u_S"', "\sim"] & \Sigma^{\infty}(\mathbb{G}_{m,T} \otimes \mathbb{Q}) \arrow[d, "u_T"'] \\
f^*(\mathbb{Q}_S(1)[1]) \arrow[r, "\sim"] & \mathbb{Q}_T(1)[1]
\end{tikzcd}$$

commutes.

Proof. By Theorem 2.3.3 in the special case $G = \mathbb{G}_{\mathrm{m}}$ (with the "Kimura dimension" $\mathrm{kd}(\mathbb{G}_{\mathrm{m}}/S)$ of the statement equal to 1), there is an isomorphism

$$\Psi := \Psi_{\mathbb{G}_{\mathrm{m}}/S} : M_S(\mathbb{G}_{\mathrm{m}}) \simeq \mathbb{Q} \oplus \Sigma^\infty(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}).$$

It is compatible with pullbacks and the isomorphisms R_f of Corollary 3.2.2 (This is the precise meaning of "compatible with pullbacks" in loc.cit). By definition, $\mathbb{Q}_S(1)[1]$ is the reduced motive of $M_S(\mathbb{G}_{\mathrm{m}})$ pointed at the unit section of $\mathbb{G}_{\mathrm{m}}$, and it follows from the naturality of $\Psi_{G/S}$ applied to the neutral section in G that the direct factor $\mathbb{Q}(1)[1]$ corresponds to the direct factor $\Sigma^\infty(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q})$. This yields an isomorphism $\widetilde{\Psi} : \mathbb{Q}_S(1)[1] \simeq \Sigma^\infty(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q})$, and we put $u_S := \widetilde{\Psi}^{-1}$. □

Remark 3.2.10. One can prove an integral refinement of Proposition 3.2.9 for S normal via a similar statement in $\mathbf{DM}^{\mathrm{eff}}$ [CDb, Proposition 11.2.11], a change of topology from Nisnevich to étale and the comparison theorem between **DM** and **DA**.

Corollary 3.2.11. *Assume S normal. Let T/S be a torus, and $X_*(T)$ its cocharacter lattice. There is an isomorphism*

$$\Sigma^\infty T_{\mathbb{Q}} \simeq \Sigma^\infty X_*(T)_{\mathbb{Q}}(1)[1].$$

In particular, the motive $\Sigma^\infty T_{\mathbb{Q}}$ is in $\mathbf{DA}^{\mathrm{gsm}}_{1,c}(S)$.

Proof. In this proof, we distinguish between derived and underived tensor products for clarity. There is a natural morphism $X_*(T) \otimes \mathbb{G}_{\mathrm{m}} \to T$ of étale sheaves on Sm/S, which is an isomorphism (this can be checked étale locally, hence for a split torus, where it is obvious). Since the functor Σ^∞ is monoidal, we have $\Sigma^\infty(X_*(T)_{\mathbb{Q}} \otimes (\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q})) \simeq \Sigma^\infty(X_*(T)_{\mathbb{Q}}) \otimes \Sigma^\infty(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}) \simeq \Sigma^\infty X_*(T)_{\mathbb{Q}}(1)[1]$ (by Proposition 3.2.9). It remains to check that the tensor product $X_*(T) \underline{\otimes} \mathbb{G}_{\mathrm{m}}$ coincides with the derived tensor product. Since S is normal, $X_*(T)_{\mathbb{Q}}$ is a direct factor of the sheaf $\mathbb{Q}(V)$ for V/S finite étale, hence it is cofibrant and we are done. This also shows that the motive in question is a direct factor of the motive of a permutation torus, and thus is geometrically smooth. □

Remark 3.2.12. This corollary probably holds for S non-normal, but we do not see how to prove it, and we only need it later for S regular. For more precise (integral) results on motives attached to tori over a field, see [HK06, §7].

We now lay the technical groundwork for the study of the motivic Picard functor in Section 3.3.2. Let $n \in \mathbb{N}$. Recall that there is an adjunction

$$\mathrm{Sus}^n : \mathbf{DA}^{\mathrm{eff}}(S) \leftrightarrows \mathbf{DA}(S) : \mathrm{Ev}_n$$

with $\mathrm{Sus}^0 = \Sigma^\infty$ and for $m \in \mathbb{N}$ a canonical isomorphism

$$\mathrm{Sus}^n(M) \simeq \Sigma^\infty M(-n)[-2n] \in \mathbf{DA}(S).$$

Using the map $u_S : \Sigma^\infty(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}) \to \mathbb{Q}_S(1)[1]$ and adjunction, we define a map

$$w_S : \mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1] \to \mathrm{Ev}_1(\mathbb{Q}_S).$$

There is an analoguous construction for motives with transfers, resulting in a map

$$w_S^{\mathrm{tr}} : \mathbb{G}_{\mathrm{m}}^{\mathrm{tr}} \otimes \mathbb{Q}[1] \to \mathrm{Ev}_1^{\mathrm{tr}}(\mathbb{Q}_S)$$

in $\mathbf{DM}^{\mathrm{eff}}(S)$.

Let $f : X \to S$ be a morphism of schemes. To state the compatibility of w_S with base change, we introduce the composition

$$d_f : f^* \mathrm{Ev}^1 \mathbb{Q}_S \xrightarrow{\epsilon} \mathrm{Ev}_1 \mathrm{Sus}^1 f^* \mathrm{Ev}_1 \mathbb{Q}_S \simeq \mathrm{Ev}_1 f^* \mathrm{Sus}^1 \mathrm{Ev}_1 \mathbb{Q}_S \xrightarrow{\eta} \mathrm{Ev}_1 f^* \mathbb{Q}_S \simeq \mathrm{Ev}_1 f^* \mathbb{Q}_X$$

where the isomorphism in the middle is the canonical isomorphism $\mathrm{Sus}^1 f^* \simeq f^* \mathrm{Sus}^1$.

Proposition 3.2.13. *Let S be a regular scheme. The morphism w_S is an isomorphism. Moreover, if $f : X \to S$ is any morphism of schemes, the following diagram*

$$\begin{tikzcd}
f^*(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1]) \arrow[r, "R_f", "\sim"'] \arrow[d, "f^*w_S"', "\sim"] & \mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1] \arrow[d, "w_X"] \\
f^* \operatorname{Ev}_1 \mathbb{Q}_S \arrow[r, "d_f"'] & \operatorname{Ev}_1 \mathbb{Q}_X
\end{tikzcd}$$

commutes, so that d_f is an isomorphism when X is also regular.

Proof. We first prove that w_S is an isomorphism. Since $\mathbf{DA}^{\mathrm{eff}}(S)$ is generated as a triangulated category by objects of the form $M_S^{\mathrm{eff}}(X)[n]$ for $f : X \to S \in \mathsf{Sm}/S$ and $n \in \mathbb{Z}$, it is enough to show that for such an object, the induced map

$$\mathbf{DA}^{\mathrm{eff}}(S)(M_S^{\mathrm{eff}}(X)[n], \mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1]) \xrightarrow{w_{S*}} \mathbf{DA}^{\mathrm{eff}}(S)(M_S^{\mathrm{eff}}(X)[n], \operatorname{Ev}_1(\mathbb{Q}_S))$$

is an isomorphism. The idea is to compare both sides to similar morphisms in the derived category $D(\mathsf{Sm}/S)$. Consider the following diagram.

$$\begin{tikzcd}
D(\mathsf{Sm}/S)(\mathbb{Q}_S(X)[n], \mathbb{G}_{\mathrm{m}}[1]) \arrow[r, "(\alpha)"] \arrow[dd, "\sim"', "\mathrm{adj}"] & \mathbf{DA}^{\mathrm{eff}}(S)(M_S^{\mathrm{eff}}(X)[n], \mathbb{G}_{\mathrm{m}}[1]) \arrow[r, "w_{S*}"] \arrow[d, "\Sigma^\infty"'] \arrow[dr, phantom, "(\mathbf{B})"] & \mathbf{DA}^{\mathrm{eff}}(S)(M_S^{\mathrm{eff}}(X)[n], \operatorname{Ev}_1(\mathbb{Q}_S)) \arrow[d, "\mathrm{adj}"', "\sim"] \\
\arrow[dr, phantom, "(\mathbf{A})"] & \mathbf{DA}(S)(M_S(X)[n], \Sigma^\infty(\mathbb{G}_{\mathrm{m}})[1]) \arrow[r, "u_{S*}", "\sim"'] \arrow[d, "\sim"', "\mathrm{adj}"] \arrow[dr, phantom, "(\mathbf{C})"] & \mathbf{DA}(S)(M_S(X)[n], \mathbb{Q}_S(1)[2]) \arrow[d, "\mathrm{adj}"', "\sim"] \\
D(\mathsf{Sm}/X)(\mathbb{Q}_X[n], f^*\mathbb{G}_{\mathrm{m}}[1]) \arrow[d, "\sim"', "R_{f*}"] & \mathbf{DA}(X)(\mathbb{Q}_X[n], f^*\Sigma^\infty\mathbb{G}_{\mathrm{m}}[1]) \arrow[r, "(f^*(u_S))_*"] \arrow[d, "\sim"', "R_{f*}"] \arrow[dr, phantom, "(\mathbf{D})"] & \mathbf{DA}(X)(\mathbb{Q}_X[n], \mathbb{Q}_X(1)[2]) \arrow[d, equal] \\
D(\mathsf{Sm}/X)(\mathbb{Q}_X[n], \mathbb{G}_{\mathrm{m}}[1]) \arrow[r] \arrow[rr, bend right, "(\beta)"'] & \mathbf{DA}(X)(\mathbb{Q}_X[n], \Sigma^\infty\mathbb{G}_{\mathrm{m}}[1]) \arrow[r, "u_{X*}", "\sim"'] & \mathbf{DA}(X)(\mathbb{Q}_X[n], \mathbb{Q}_X(1)[2])
\end{tikzcd}$$

The square (A) commutes because the isomorphisms R_f in the derived category and in $\mathbf{DA}$ are compatible by construction. The square (B) commutes by construction of w_S and u_S. The square (C) commutes by naturality of adjunction. Finally, the square (D) commutes by Proposition 3.2.9.

To complete the proof that w_{S*} is an isomorphism, it remains to see that the maps (α) and (β) are isomorphisms as well. For (β), this is precisely the statement of Proposition 3.B.4 (ii)-(iv). Let us prove that (α) is an isomorphism.

Since S is regular, all smooth S-schemes are regular. They are in particular reduced, which implies that $\mathbb{G}_{\mathrm{m}}$ is $\mathbb{A}^1$-invariant on Sm/S, and normal, which implies that $\mathrm{Pic} = H^1(-, \mathbb{G}_{\mathrm{m}})$ is $\mathbb{A}^1$-invariant. The higher cohomology groups $H^i(-, \mathbb{G}_{\mathrm{m}})$ for $i \geq 2$ are torsion on regular schemes by [Gro68, Proposition 1.4]. All together, this implies that the sheaf $\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}$ is $\mathbb{A}^1$-local in the model category underlying $\mathbf{DA}^{\mathrm{eff}}(S)$. This implies that the morphism $(\alpha) : \mathsf{D}(\mathsf{Sm}/S)(\mathbb{Q}_S(X)[n], \mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1]) \to \mathbf{DA}^{\mathrm{eff}}(M_S^{\mathrm{eff}}(X), \mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q})$ is an isomorphism. This completes the proof that w_S is an isomorphism.

It remains to adress the commutation of the diagram in the statement. Going through the definitions of w_S and d_f, we see that it is obtained from the commutative diagram of Proposition 3.2.9 via the adjunction $\mathrm{Sus}^1 \dashv \operatorname{Ev}_1$ and the commutation of Sus^1 and f^*. $\square$

We come to the fundamental property of $\mathbf{DA}_{\mathrm{gr}}(S)$ from the point of view of this paper: it is a source of compact homological 1-motives.

Proposition 3.2.14. *Let $M \in \mathbf{DA}_{\mathrm{gr}}(S)$. Then M lies in $\mathbf{DA}_{1,c}(S)$.*

Proof. We can assume M is of the form $M = \Sigma^\infty G \otimes \mathbb{Q}$. By [AHPL14, Theorem 3.3.(3)], M is a compact motive.

It remains to show that M is an homological 1-motive. The proof of [AHPL14, Theorem 3.3.(3)] essentially shows this as well, but we provide an argument for convenience. By compactness and Proposition 3.1.24, it is enough to show that for all $s \in S$, $s^*\Sigma^\infty M$ is in $\mathbf{DA}_1(s)$. By Proposition 3.2.4 (in the case $K_* = G[0]$), continuity for $\mathbf{DA}_1(-)$ (Proposition 3.1.22) and separation, we are reduced to the case where S is the spectrum of a perfect field k.

The group scheme G over the field k has a neutral component G° which is smooth and of finite type. The quotient group scheme G/G° is a discrete group scheme so its motive lies in $\mathbf{DA}_0(k) \subset \mathbf{DA}_1(k)$. In the case of a smooth commutative connected algebraic group, we reduce by structure theory to the case of unipotent algebraic groups, tori and abelian varieties.

A unipotent algebraic group over a perfect field is $\mathbb{A}^1$-contractible. If $G = T$ is a torus, let $e : \mathbf{Spec}(l) \to \mathbf{Spec}(k)$ be a finite étale morphism with T_l split. Then $e^*\Sigma^\infty(T \otimes \mathbb{Q}) \simeq \Sigma^\infty(T_l \otimes \mathbb{Q})$ (this is easy because e is smooth). By a transfer argument using [Ayo07a, Lemme 2.1.165] and Proposition 3.1.14 (ii), this reduces us to the case of split tori, and then by direct sum to the case of $\mathbb{G}_m$, which follows from Proposition 3.2.9. If $G = A$ is an abelian variety, using [Kat99, Theorem 11] reduces the case of A to the case of a Jacobian $J(C)$ of a smooth projective curve C/k with a rational point. The fact that $\Sigma^\infty(J(C) \otimes \mathbb{Q})$ is in $\mathbf{DA}_1(k)$ follows from Proposition 3.2.8. □

3.2.2 Motives of Deligne 1-motives

We relate the classical category $\mathcal{M}_1(S)$ of Deligne 1-motives (recalled in Section 3.A) to our setup of $\mathbf{DA}(S)$.

Notation 3.2.15. Let K_* be a bounded complex of commutative group schemes over S. We use the same notation $K_* \otimes \mathbb{Q}$ for the induced complex of sheaves of $\mathbb{Q}$-vector spaces on $(\mathsf{Sm}/S)_{\text{ét}}$, the corresponding object in the derived category, and the induced effective motive in $\mathbf{DA}^{\text{eff}}(S)$. We write $\Sigma^\infty(K_* \otimes \mathbb{Q})$ for the T-suspension spectrum built on $K_* \otimes \mathbb{Q}$, and for the induced motive in $\mathbf{DA}(S)$. We denote the induced functor on Deligne 1-motives by

$$\mathcal{R} = \mathcal{R}_S : \mathcal{M}_1(S) \to \mathbf{DA}_1(S), \quad \mathbb{M} \mapsto \Sigma^\infty(\mathbb{M} \otimes \mathbb{Q}).$$

Corollary 3.2.16. *Let $\mathbb{M} = [L \to G] \in \mathcal{M}_1(S)$. Then $\mathcal{R}(\mathbb{M})$ lies in $\mathbf{DA}_{1,c}(S)$. If S is moreover assumed to be normal, then the motive $\mathcal{R}(\mathbb{M})$ is also geometrically smooth, thus lies in $\mathbf{DA}^{\text{gsm}}_{1,c}(S)$.*

Proof. We apply Proposition 3.2.14 to the distiguished triangle

$$\Sigma^\infty G_{\mathbb{Q}}[-1] \to \mathcal{R}(\mathbb{M}) \to \Sigma^\infty L_{\mathbb{Q}} \xrightarrow{+}$$

which proves the first part. Assume now S to be normal. We have a further distinguished triangle

$$\Sigma^\infty T_{\mathbb{Q}} \to \Sigma^\infty G_{\mathbb{Q}} \to \Sigma^\infty L_{\mathbb{Q}} \xrightarrow{+} .$$

The motives $\Sigma^\infty T_{\mathbb{Q}}$ and $\Sigma^\infty L_{\mathbb{Q}}$ are geometrically smooth by Corollary 3.2.11 and its proof. The motive $\Sigma^\infty A_{\mathbb{Q}}$ is a direct factor of the homological motive of A by Theorem 2.3.3, so it is geometrically smooth. This completes the proof. □

From Corollary and the definition of $\mathcal{R}$, we deduce the following.

Corollary 3.2.17. *Let $f : T \to S$ be a morphism of schemes. There is an isomorphism of functors*

$$R_f : f^*\mathcal{R}_S \xrightarrow{\sim} \mathcal{R}_T f^*.$$

which is compatible with further pullbacks.

3.2.3 Picard complex

Classically the Picard functor of a morphism of schemes f is defined as $R^1 f_* \mathbb{G}_{\mathrm{m}}$. We introduce a variant of this construction which includes information about relative connected components.

Definition 3.2.18. Let $f : X \to S$ be a morphism of schemes. The Picard complex $\mathrm{P}(X/S)$ of X over S is the object $\tau_{\geq 0} f_*(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1]) \in D_{[0,1]}(\mathsf{Sm}/S)$.

Remark 3.2.19. Recall from [SGA73, Exposé XVIII §1.4] that there is an equivalence of categories between the category of commutative group stacks over a site $\mathcal{S}$ (with morphisms taken up to 2-isomorphisms) and the category $D_{[0,1]}(\mathbf{Sh}(\mathcal{S}, \mathbb{Z}))$. The Picard complex corresponds via this equivalence to the smooth Picard stack, i.e., the version for Sm/S of the usual Picard stack (see e.g. [Bro09]). This point of view will not be used explicitly in the rest of this paper.

We will also need a version with transfers.

Definition 3.2.20. Let S be a scheme, $f : X \to S$ a morphism of schemes. The Picard complex with transfers $\mathrm{P}^{\mathrm{tr}}(X/S)$ of X over S is the object $\tau_{\geq 0} f_*(\mathbb{G}_{\mathrm{m}}^{\mathrm{tr}} \otimes \mathbb{Q}[1]) \in D_{[0,1]}(\mathrm{Cor}/S)$. There is a canonical map

$$a^{\mathrm{tr}} P(X/S) \longrightarrow P^{\mathrm{tr}}(X/S)$$

coming from adjunction and Proposition 2.2.10 for $\mathbb{G}_{\mathrm{m}}$.

We proceed to analyse the structure of $\mathrm{P}(X/S)$, following closely the standard structure theory for the Picard scheme [Kle05] and the Picard stack [Bro09]. We will see that restricting to the smooth site discards quite a bit of information and leads to simpler structure.

In the sequel, we consider étale sheaves of abelian groups and $\mathbb{Q}$-vector spaces on the two sites Sch/S and Sm/S. We have a morphism of sites $\zeta : \mathsf{Sch}/S \to \mathsf{Sm}/S$. The restriction functor $\zeta_* : \mathbf{Sh}(\mathsf{Sch}/S) \to \mathbf{Sh}(\mathsf{Sm}/S)$ is exact since both sites have the same points. We have $\zeta_* \mathbb{G}_{\mathrm{m}} \simeq \mathbb{G}_{\mathrm{m}}$. The functor ζ_* commutes with f_* and $\underline{f}_*$. By abuse of terminology, we will say that a sheaf of sets on Sm/S is representable if it is isomorphic to the functor $\zeta_* X$ for X a not-necessarily smooth S-scheme; such a scheme is then not uniquely determined up to isomorphism.

To study $\mathrm{P}(X/S)$, we restrict to the following situation.

Hypothesis 3.2.21. Let S be a noetherian scheme and $f : X \to S$ be a smooth projective morphism.

This has some useful consequences. First, by [Gro67, 17.16.3 (ii)], the morphism f has sections locally in the étale topology. Second, by [Gro63, 7.8.6], the morphism f has a Stein factorisation

$$f : X \xrightarrow{f^\circ} \pi_0(X/S) := \mathbf{Spec}_S(\underline{f}_* \mathcal{O}_X) \xrightarrow{\pi_0(f)} S$$

with $\pi_0(f)$ finite étale and the construction of $\underline{f}_* \mathcal{O}_X$ (and hence $\pi_0(X/S)$) commutes with arbitrary base change, i.e., f is cohomologically flat in degree 0. Notice that in many treatments of the Picard scheme, the stronger hypothesis "$\underline{f}_* \mathcal{O}_X \simeq \mathcal{O}_S$ universally" is used, but that here we want to keep track of the relative connected components.

Since we are only interested in the rational coefficient situation, we have the following simplification.

Lemma 3.2.22. *Let $f : X \to S$ be a smooth morphism with S regular. Then for $i \geq 1$, the sheaf $R^i f_*(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1]) \simeq R^{i+1}(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q})$ is trivial. As a consequence, we have*

$$\mathrm{P}^{(\mathrm{tr})}(X/S) \xrightarrow{\sim} f_*(\mathbb{G}_{\mathrm{m}}^{\mathrm{tr}} \otimes \mathbb{Q}[1]).$$

Proof. This follows from the fact that for a regular scheme T and $i \geq 2$, the étale cohomology groups $H^i(T, \mathbb{G}_{\mathrm{m}})$ are torsion [Gro68, Proposition 1.4]. □

We first look at the sheaf $\underline{f}_*(\mathbb{G}_{\mathrm{m}})$. For any $U \to S$ smooth, we have $\underline{f}_*(\mathbb{G}_{\mathrm{m}})(U) = \mathcal{O}^\times(X \times_S U) \simeq \mathcal{O}^\times(\pi_0(X \times U/U)) \simeq \mathcal{O}^\times(\pi_0(X/S) \times_S U)$. This shows that $\underline{f}_* \mathbb{G}_{\mathrm{m}}$ is representable by a torus, the Weil restriction $\mathrm{Res}_{\pi_0(f)} \mathbb{G}_{\mathrm{m}}$ (see Definition 3.A.11).

Next, we look at the classical Picard étale sheaf $\mathcal{P}ic_{X/S} := R^1 f_*\mathbb{G}_{\mathrm{m}} \in \mathbf{Sh}((\mathsf{Sch}/S)_{\text{ét}}, \mathbb{Z})$ and its smooth analogue $\mathcal{P}ic^{\mathrm{sm}}_{X/S} \in \mathbf{Sh}((\mathsf{Sm}/S)_{\text{ét}}, \mathbb{Z})$ defined by the same formula on the smooth site. By exactness of ζ_*, we have $\zeta_*\mathcal{P}ic_{X/S} \simeq \underline{\zeta}_*\mathcal{P}ic_{X/S} \simeq \mathcal{P}ic^{\mathrm{sm}}_{X/S}$.

Because f has sections locally in the étale topology, as a corollary of the Leray spectral sequence, we have for all $T \in \mathsf{Sm}/S$ a short exact sequence

$$0 \to \mathrm{Pic}(\pi_0(X_T/T)) \to \mathrm{Pic}(X_T) \to \mathcal{P}ic^{(\mathrm{sm})}_{X/S}(T) \to 0. \tag{L}$$

The functors $\mathcal{P}ic_{X/S}$ come with natural subfunctors $\mathcal{P}ic^0_{X/S}$ and $\mathcal{P}ic^\tau_{X/S}$, the neutral component and the torsion component (i.e., elements "with a multiple in the neutral component"), which are special cases of the following general definition.

Definition 3.2.23. Let G be a functor $(\mathsf{Sch}/S)^{\mathrm{op}} \to \mathrm{Ab}$. We define two group subfunctors G° and G^τ as follows. If $T = \mathbf{Spec}(k) \to S$ is the spectrum of an algebraically closed field k, then a point $t \in G(T)$ is in $G^0(T)$ if t is algebraically equivalent to 0 in the natural sense (i.e it can be connected to the neutral section of G by a sequence of smooth connected k-curves). The point t is in $G^\tau(T)$ iff there exists $n > 0$ with $t^n \in G^\circ(T)$. If X is a general object in $\mathcal{S}$, a point $t \in G(X)$ is in $G^0(X)$ (resp. $G^\tau(X)$) iff for all morphisms $\xi : T = \mathbf{Spec}(k) \to X$ for k algebraically closed, the restriction $\xi^*(t)$ is in $G^0(T)$ (resp. in $G^\tau(T)$).

We then define $\mathcal{P}ic^{\mathrm{sm},0}_{X/S}$ (resp. $\mathcal{P}ic^{\mathrm{sm},\tau}_{X/S}$) as $\underline{\zeta}_*\mathcal{P}ic^0_{X/S}$(resp. $\underline{\zeta}_*\mathcal{P}ic^\tau_{X/S}$). The following is easy and well-known for $\mathcal{P}ic$; the proof translates directly to $\mathcal{P}ic^{\mathrm{sm}}$ and $P(X/S)$.

Lemma 3.2.24. *Let $\pi : T \to S$ a morphism of schemes. There is a natural isomorphism*

$$v_\pi : \pi^*\mathcal{P}ic_{X/S} \simeq \mathcal{P}ic_{X\times_S T/T}$$

which respects the neutral and the torsion components and which is compatible with further pullbacks and the isomorphisms R_g (i.e., a diagram like the one in Proposition 3.2.9 commutes). Similarly, there is natural morphism

$$v_\pi : \pi^*\mathcal{P}ic^{sm}_{X/S} \to \mathcal{P}ic^{sm}_{X\times_S T/T},$$

(resp.

$$v_\pi : \pi^*\mathrm{P}(X/S) \simeq \mathrm{P}(X \times_S T/T))$$

which is an isomorphism when π is smooth and is compatible with further pullbacks in the same way.

In general, the construction of $\mathcal{P}ic^{\mathrm{sm}}$ and $P(X/S)$ does not commute with arbitrary base change, i.e., v_π is not always an isomorphism. We will see below some positive results.

We recall the following classical positive results of Grothendieck on the Picard scheme. We state the result both for the classical and the smooth context; the smooth result follows immediately by applying $\underline{\zeta}_*$.

Theorem 3.2.1. *Under Hypothesis 3.2.21, the following statements hold.*

(i) *[Gro95a, Theoreme 3.1] The functor $\mathcal{P}ic^{(\mathrm{sm})}_{X/S}$ is representable by a commutative group scheme, locally of finite type over S, that we we denote $\mathrm{Pic}^{(sm)}_{X/S}$.*

(ii) *[Gro95b, Corollaire 2.3] The functor $\mathcal{P}ic^{(\mathrm{sm}),\tau}_{X/S}$ is representable by a projective group scheme $\mathrm{Pic}^\tau_{X/S}$, which is an open and closed group subscheme of $\mathrm{Pic}_{X/S}$.*

For the neutral component, the situation is more complex. We have nevertheless positive results that will be enough for us.

Theorem 3.2.2. *Under hypothesis 3.2.21, the following holds.*

(i) *[Gro95b, Corollaire 3.2] If S is the spectrum of a field k, then $\mathrm{Pic}^0_{X/k}$ is representable by a projective algebraic group, with $\mathrm{Pic}^{0,\mathrm{red}}_{X/k} := (\mathrm{Pic}^0_{X/k})^{\mathrm{red}}$ an abelian variety.*

(ii) *[Bro14, Proposition 2.15] If* $\mathrm{Pic}^{\tau}_{X/S}$ *is flat and the construction of* $f_*\mathcal{O}_X$ *commutes with base change, then* $\mathrm{Pic}^{\tau}_{X_S/S}$ *is an extension of finite flat commutative group scheme by an abelian scheme. We call* $\mathrm{Pic}^{0,\mathrm{red}}_{X/S}$ *this abelian scheme. We use the following notation for this exact sequence.*

$$0 \to \mathrm{Pic}^{0,\mathrm{red}}_{X/S} \to \mathrm{Pic}^{\tau}_{X/S} \to F \to 0$$

(iii) *If* S *is the spectrum of a field* k*, the condition of* (ii) *holds and the two abelian varieties* $\mathrm{Pic}^{0,\mathrm{red}}_{X/k}$ *defined above coincide.*

Proof. The only thing to prove is (iii). Flatness and cohomological flatness automatically hold over a field. Let $G = \mathrm{Pic}^{\tau}_{X/k}$ which is a commutative algebraic group with neutral component $G^0 = \mathrm{Pic}^0_{X/k}$. By (i) we have $A := G^0_{\mathrm{red}}$ abelian variety. By (ii) we have $0 \to B \to G \to F \to 0$ with B abelian variety and F a finite flat group scheme. Both A and B are sub-abelian varieties of G and we must show $A = B$. Since $\mathrm{Hom}(A, F) = 0$ because A is connected and reduced, we have $A \subset B$. Since B is connected and reduced, we have $B \subset A$. We conclude that $A = B$, as required. $\square$

Definition 3.2.25. We say that $f : X \to S$ is Pic-smooth if $\mathrm{Pic}^{\tau}_{X/S}$ is flat and cohomologically flat in degree 0.

Remark 3.2.26. By Lemma 3.2.24 and the fact that flatness and cohomological flatness are stable by arbitrary base change, we see that Pic-smoothness is also stable by arbitrary base change.

Proposition 3.2.27. *Let* $f : X \to S$ *be a smooth projective morphism. Assume* S *is reduced. Then there is a dense open set* $U \subset S$ *such that* $f \times_S U$ *is Pic-smooth.*

Proof. Recall that $\mathrm{Pic}^{\tau}_{X/S}$ is representable by a group scheme of finite type by Theorem 3.2.1 (ii). If S is reduced, generic flatness [Gro65, Corollaire 6.9.3] provides a dense open subset V of S over which $\mathrm{Pic}^{\tau}_{X/S} \times_S U \simeq \mathrm{Pic}^{\tau}_{X_V/V}$ is flat.

By standard results on cohomology and base change [Gro63, §7], because V is reduced, the flat morphism $\pi : P = \mathrm{Pic}^{\tau}_{X_V/V} \to V$ is cohomologically flat in dimension 0 if the function $d^1 : s \in V \to H^1(P_s, \mathcal{O}_s)$ is locally constant. The function d^1 is upper semi-continuous [Gro63, Theorem 7.7.5, I], hence, locally constant on a dense open set U of S. This implies that $f \times_S U$ is Pic-smooth. $\square$

Proposition 3.2.28. *Let* $f : X \to S$ *be a smooth projective Pic-smooth morphism. Then* $\mathcal{P}ic^{\mathrm{sm},0}_{X/S}$ *is representable by the abelian scheme* $\mathrm{Pic}^{0,\mathrm{red}}_{X/S}$.

Proof. By Theorem 3.2.2 (ii), we have a short exact sequence of group schemes

$$0 \to \mathrm{Pic}^{0,\mathrm{red}}_{X/S} \to \mathrm{Pic}^{\tau}_{X/S} \to F \to 0$$

with F finite flat group scheme. Let $T \in \mathsf{Sm}/S$ and $L \in \mathcal{P}ic^{\mathrm{sm},0}_{X/S}(T) \subset \mathrm{Pic}^{\tau}_{X/S}(T)$. Let $\xi : \mathbf{Spec}(k) \to T$ be a geometric point. By hypothesis, there exists a sequence of smooth curves connecting $\xi^*(L)$ to the zero section of $\mathrm{Pic}^{\tau}_{X_k/k}$. Any morphism from a smooth curve over k to F_k is constant. This shows that the image of $\xi^*(L)$ in $F(k)$ is zero. Since this holds for all k and T/S is smooth, this implies that the induced morphism $T \to F$ is zero. This shows that we have an induced monomorphism $\mathcal{P}ic^{\mathrm{sm},0}_{X/S} \to \mathrm{Pic}^{0,\mathrm{red}}_{X/S}$.

In the other direction, the morphism $\mathrm{Pic}^{0,\mathrm{red}}_{X/S} \to \mathrm{Pic}^{\tau,\mathrm{sm}}_{X/S}$ factors through $\mathcal{P}ic^{\mathrm{sm},0}_{X/S}$ because abelian varieties are geometrically connected. This shows the above monomorphism is surjective and concludes the proof. $\square$

We now turn to the study of the Néron-Severi groups in families.

Definition 3.2.29. We define the Neron-Severi sheaf as the étale quotient sheaf

$$\mathcal{NS}^{(\mathrm{sm})}_{X/S} := \mathcal{P}ic^{(\mathrm{sm})}_{X/S} / \mathcal{P}ic^{(\mathrm{sm}),\tau}_{X/S}.$$

The following lemma follows immediately from the properties of $\mathcal{P}ic$ above.

Lemma 3.2.30. *We have a canonical isomorphism $\zeta_*\mathcal{NS}_{X/S} \simeq \mathcal{NS}^{\mathrm{sm}}_{X/S}$, and the construction of $\mathcal{NS}_{X/S}$ (resp. $\mathcal{NS}^{\mathrm{sm}}_{X/S}$) commutes with base change by an arbitrary morphism (resp. by a smooth morphism).*

Lemma 3.2.31. *Let $f : X \to S$ be Pic-smooth with S regular. Then for all $T \in \mathsf{Sm}/S$, we have*

$$\mathcal{NS}^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q}(T) \simeq \mathcal{P}ic^{\mathrm{sm}}_{X/S}(T) \otimes \mathbb{Q}/\mathcal{P}ic^{\mathrm{sm},\tau}_{X/S}(T) \otimes \mathbb{Q}.$$

Proof. It suffices to prove that in this situation, the cohomology group $H^1_{\text{ét}}(T, \mathcal{P}ic^{\mathrm{sm},\tau}_{X/S} \otimes \mathbb{Q})$ vanishes. By Theorem 3.2.2 (ii), we have a short exact sequence

$$0 \to \mathrm{Pic}^{0,\mathrm{red}}_{X/S} \to \mathrm{Pic}^{\tau}_{X/S} \to F \to 0$$

where $\mathrm{Pic}^{0,\mathrm{red}}$ is an abelian scheme and F is a finite flat commutative group scheme. Almost by definition, classes in $H^1(T, F)$ can be trivialized by passing to a finite flat cover, so by a transfer argument they are torsion and hence vanish after tensoring by $\mathbb{Q}$. On the other hand, since T is noetherian and regular, [Ray70b, Proposition XIII 2.6.(ii)] and [Ray70b, Proposition XIII 2.3.(ii)] imply that torsors under $\mathrm{Pic}^{0,\mathrm{red}}$ are torsion, which implies that $H^1(T, \mathrm{Pic}^{0,\mathrm{red}} \otimes \mathbb{Q}) = 0$. This concludes the proof. $\square$

We have a morphism of sites $\alpha : \mathsf{Sm}/S \to \mathrm{Et}/S$ where Et/S is the small étale site of S. Put $\gamma = \alpha \circ \zeta : \mathsf{Sch}/S \to \mathrm{Et}/S$. We say that a sheaf F on Sm/S (resp. Sch/S) is constructible if it is in the essential image of the fully faithful functor α^* (resp. γ^*), or equivalently if the counit morphism $\alpha^*\alpha_*F \to F$ (resp. $\gamma^*\gamma_*F \to F$) is an isomorphism and α_*F (resp. γ_*F) is constructible (as a sheaf of $\mathbb{Z}$-modules, i.e., we do not require fibers to be finite abelian groups, but only to be finitely generated).

It is well known that the sheaf $\mathcal{NS}_{X/S}$ is far from being constructible; in particular, the rank of the geometric fibers (which are finitely generated abelian groups by [SGA71, Exp XIII, Thm 5.1]) is not a constructible function [BLR90b, 8.4 Remark 8]. For the smooth Neron-Severi sheaf, the situation is somewhat better.

Proposition 3.2.32. *Let $f : X \to S$ be Pic-smooth with S regular. The sheaf $\mathcal{NS}^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q}$ is locally constant.*

Proof. We can assume S is connected, with generic point η. Fix a geometric point $\bar{\eta}$ over η. Since being locally constant is an étale local property and f is smooth, we can assume that f has a section s.

By [SGA71, Exp XIII, Thm 5.1], the abelian group $\mathrm{NS}(X_{\bar{\eta}})$ is finitely generated. It comes with a continuous action of the Galois group $\mathrm{Gal}(\bar{\eta}/\eta)$ (which thus factors through a finite quotient). The ℓ-adic first Chern class yield a Galois-equivariant morphism $c_1 : \mathrm{NS}(X_{\bar{\eta}}) \to H^2(X_{\bar{\eta}}, \mathbb{Q}_l(1))$ which is injective after tensoring by $\mathbb{Q}_l$, hence also injective after tensoring by $\mathbb{Q}$. Moreover, since f is smooth and projective, for any codimension 1 point $s \in S$, the Galois representation on $H^2(X_{\bar{\eta}}, \mathbb{Q}_l(1))$ is unramified at s. This implies that $\mathrm{NS}(X_{\bar{\eta}})_{\mathbb{Q}}$ is also unramified at s. Since S is regular, this equips $\mathrm{NS}(X_{\bar{\eta}})_{\mathbb{Q}}$ with an action of the étale fundamental group of S at $\bar{\eta}$, which is none other than the unramified quotient of $\mathrm{Gal}(\bar{\eta}/\eta)$ [SGA03, Proposition 8.2]. This implies that $\mathrm{NS}(X_{\bar{\eta}})_{\mathbb{Q}}$ can be identified with the geometric generic fiber of a locally constant constructible étale sheaf of $\mathbb{Q}$-vector spaces $\mathcal{N}_{X/S}$, the *Neron-Severi lattice* of X over S.

We now define a morphism $e_S : \alpha_*\mathcal{NS}^{\mathrm{sm}}_{X/S} \to \mathcal{N}_{X/S}$ as follows. We first define a morphism $\tilde{c}_S : \alpha_*\mathrm{Pic}^{\mathrm{sm}}_{X/S} \to \mathcal{N}_{X/S}$. Recall that $\mathrm{Pic}^{\mathrm{sm}}_{X/S}$ is the étale sheaf associated to the presheaf $\mathrm{Pic}^{\mathrm{sm},\mathrm{psh}}_{X/S}$: $V \in \mathrm{Et}/S \mapsto \mathrm{Pic}(X \times_S U)$. Since $\mathcal{N}_{X/S}$ is an étale sheaf, defining $\tilde{c}_S$ is equivalent to writing down a morphism $\mathrm{Pic}^{\mathrm{sm},\mathrm{psh}}_{X/S} \to \mathcal{N}_{X/S}$. Let $V \in \mathrm{Et}/S$, which we can assume connected, and $\mathcal{L}$ be a line bundle on $X \times_S V$. Choose a factorisation $\bar{\eta} \to V_\eta \to \eta$, which induces a morphism $\pi_1(V, \bar{\eta}) \to \pi_1(S, \bar{\eta})$. Using this factorisation, lift $\mathcal{L}_\eta$ to a class in $\mathrm{NS}(X_{\bar{\eta}}) \simeq \mathrm{NS}(X_{V_\eta} \times_{V_\eta} \bar{\eta})$ which by construction is fixed by $\pi_1(V, \bar{\eta})$, so gives a section in $\mathcal{N}_{X/S}(V)$. This is the required class $\tilde{c}_S([\mathcal{L}])$. The morphism $\tilde{c}_S$ is trivial on $\alpha_*\mathrm{Pic}^{\mathrm{sm},\tau}_{X/S}$ since algebraic equivalence over V implies algebraic equivalence over $\bar{\eta}$. So $\tilde{c}_S$ induces a morphism $e_S : \alpha_*\mathcal{NS}^{\mathrm{sm}}_{X/S} \to \mathcal{N}_{X/S}$ as required.

The goal of the rest of the proof is to establish that

a) the counit morphism $\alpha^*\alpha_*\mathcal{NS}^{\mathrm{sm}}_{X/S} \longrightarrow \mathcal{NS}^{\mathrm{sm}}_{X/S}$ is an isomorphism, and

b) $e_S : \alpha_*\mathcal{NS}^{\mathrm{sm}}_{X/S} \to \mathcal{N}_{X/S}$ is an isomorphism,

which imply the proposition. We want to reduce the proof of points a) and b) to the case where S is a field, by restriction to the generic point η.

Lemma 3.2.33. *With the hypotheses of the proposition, assume S is moreover irreducible and denote by η the generic point of S. Then the adjunction morphism*

$$\mathcal{NS}^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q} \to \eta_*\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q}$$

is an isomorphism.

Proof. Since η is pro-smooth, we deduce from Lemma 3.2.30 that $\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S} \simeq \mathcal{NS}^{\mathrm{sm}}_{X_\eta/\eta}$.

Let $T \in \mathsf{Sm}/S$. One sees that the map $\mathcal{NS}^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q}(T) \to \eta_*\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q}(T)$ can be identified, modulo the previous paragraph and the identification of Lemma 3.2.31 with the natural map

$$\mathcal{P}ic_{X/S}(T)_{\mathbb{Q}}/\mathcal{P}ic^{\tau}_{X/S}(T)_{\mathbb{Q}} \longrightarrow \mathcal{P}ic_{X_\eta/\eta}(T_\eta)_{\mathbb{Q}}/\mathcal{P}ic^{\tau}_{X_\eta/\eta}(T_\eta)_{\mathbb{Q}}$$

which because f has a section takes the more concrete form

$$\mathrm{Pic}(X_T)_{\mathbb{Q}}/(f_T^*\mathrm{Pic}(T)_{\mathbb{Q}}\mathcal{P}ic^{\tau}_{X/S}(T)_{\mathbb{Q}}) \longrightarrow \mathrm{Pic}(X_{T_\eta})_{\mathbb{Q}}/(f_{T_\eta}^*\mathrm{Pic}(T_\eta)\mathcal{P}ic^{\tau}_{X/S}(T_\eta)_{\mathbb{Q}})$$

We need to show that this map is bijective. The surjectivity follows immediately from the fact that, since T is regular, isomorphism classes of lines bundles can be represented by Weil divisors. We prove the injectivity. Let $\mathcal{L} \in \mathrm{Pic}(X_T)$ such that $\mathcal{L}_\eta \in \mathrm{Pic}(X_{T_\eta})$ lies in $f_{T_\eta}^*\mathrm{Pic}(T_\eta)\mathcal{P}ic^{\tau}_{X/S}(T_\eta)$. One can find a dense open set $U \subset S$ such that $\mathcal{L}_U$ lies in $f_{T_U}^*\mathrm{Pic}(T_U)\mathcal{P}ic^{\tau}_{X/S}(T_U)$, say $\mathcal{L}_U = f_{T_U}^*\mathcal{L}' \cdot x^\tau$. Because $\mathcal{P}ic^{\tau}_{X/S}$ is an abelian scheme "up to a finite flat group scheme" (by the Pic-smooth hypothesis, Theorem 3.2.2 (ii)) and we work tensor $\mathbb{Q}$, we can use the flasqueness of abelian schemes over regular schemes (see e.g. [Bha12, Proposition 4.2]) to extend x^τ to an element of $\mathcal{P}ic^{\tau}_{X/S}(T)$. We can thus assume that x^τ is trivial and that $\mathcal{L}_U = f_{T_U}^*\mathcal{L}'$. By using the argument for surjectivity on T for $\mathcal{L}'$, we see that we can also assume that $\mathcal{L}'$ is trivial. We are thus reduced to the case where $\mathcal{L}_{T_U}$ is trivial.

Let D be a Weil divisor on X_T such that $\mathcal{L} \simeq \mathcal{O}_{X_T}(D)$. By the above, $D \cap T_U = \emptyset$. Because D is a divisor and f_T has connected fibers, this implies that D contains every fiber of f_T it meets, which shows that D comes from a divisor on T. So $\mathcal{L}$ is in $f_T^*\mathrm{Pic}(T)$ and the injectivity is proven. $\square$

Consider the following diagram of morphisms of sites

$$\begin{tikzcd}
\mathsf{Sm}/\eta \arrow[r, "\beta"] \arrow[d, "\eta"'] & \mathrm{Et}/\eta \arrow[d, "\gamma"] \\
\mathsf{Sm}/S \arrow[r, "\alpha"'] & \mathrm{Et}/S.
\end{tikzcd}$$

Since a smooth S-scheme which is étale over η is étale over a dense open set, we see that the base change morphism $\alpha^*\gamma_* \longrightarrow \eta_*\beta^*$ is an isomorphism. We have a commutative diagram

$$\begin{tikzcd}
\alpha^*\alpha_*\mathcal{NS}^{\mathrm{sm}}_{X/S} \arrow[r, "\sim"] \arrow[d] & \alpha^*\alpha_*\eta_*\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S} \arrow[r, "\sim"] \arrow[d] & \alpha^*\gamma_*\beta_*\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S} \arrow[r, "\sim"] & \eta_*\beta^*\beta_*\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S} \arrow[d] \\
\mathcal{NS}^{\mathrm{sm}}_{X/S} \arrow[r, "\sim"] & \eta_*\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S} \arrow[rr, equal] & & \eta_*\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S}
\end{tikzcd}$$

where all the maps come from the adjunctions and the commutativity is formal. The isomorphisms in the first square come from Lemma 3.2.33. We have $\eta^*\mathcal{NS}^{\mathrm{sm}}_{X/S} \simeq \mathcal{NS}^{\mathrm{sm}}_{X_\eta/\eta}$ by Lemma 3.2.30; this shows that to prove point a) above, one can assume $S = \eta$. A similar argument, using the fact that the construction of e_S commute with restriction to η and that $\mathcal{N}_{X/S}$ is a lattice (so that it satisfies $\mathcal{N}_{X/S} \simeq \gamma_*\gamma^*\mathcal{N}_{X/S} \simeq \gamma_*\mathcal{N}_{X_\eta/\eta}$), shows that to prove point b) one can assume $S = \eta$ as well.

This reduces the proof of a) and b) to the case where S is the spectrum of a field k. A Galois descent argument then shows that we can assume k is separably closed, so that the sheaf $\mathcal{N}_{X/S}$ becomes constant.

A convenient feature of the field case is that the morphism of sites $\alpha : \mathsf{Sm}/S \to \mathrm{Et}/S$ admits a section $\pi_0 : \mathrm{Et}/S \to \mathsf{Sm}/S$, the functor which associates to a smooth k-scheme U the étale k-scheme $\pi_0(U/k) := \mathbf{Spec}(\mathcal{O}_U(U))$; moreover, we have a canonical isomorphism $\alpha^* \simeq \pi_{0*}$.

We will prove a) and b) by examining the induced maps on points of the smooth étale site $(\mathsf{Sm}/k)_{\text{ét}}$. Let U be a smooth k-scheme and $\bar{x} \to U$ be a geometric point. Let $V = U_{\bar{x}}^{\mathrm{hs}}$ be the strict henselisation of U at $\bar{x}$, considered as a smooth pro-k-scheme $\mathbf{Spec}(R)$. Then the collection of all such V gives enough point of the site Sm/k. Moreover, applying π_0, we get an étale k-scheme $\pi_0(V)$ with a map $V \to \pi_0(V)$; here $\pi_0(V)$ is the spectrum of the separable closure $\tilde{k}$ of k in the k-algebra R. Since R is noetherian, regular and local, it is factorial, so $\mathrm{Pic}(V) = 0$. Using the relationship of $\mathrm{Pic}(X \times_k V)$ with Weil divisors as in the proof of Lemma 3.2.33, we can show that the map

$$\mathcal{NS}^{\mathrm{sm}}_{X/k}(V) \simeq \mathrm{Pic}(X \times_k V)/\mathcal{P}ic^{\tau}_{X/k}(V) \to \mathrm{Pic}(X \times_k \pi_0(V))/\mathcal{P}ic^{\tau}_{X/k}(\pi_0(V)) \simeq \mathcal{NS}^{\mathrm{sm}}_{X/k}(\pi_0(V))$$

is an isomorphism, proving a). Similarly $\mathrm{Pic}(\pi_0(V)) = 0$. From this, one deduces that the group $\mathcal{NS}^{\mathrm{sm}}_{X/k}(\pi_0(V))$ is isomorphic to the Neron-Severi group $\mathrm{NS}(X_{\bar{x}})$. Combined with a), this proves b) and completes the proof. $\square$

This result has several useful corollaries.

Corollary 3.2.34. *Assume S is regular. Let $f : X \to S$ be a smooth projective Pic-smooth morphism of schemes. Then the smooth Picard complex $\mathrm{P}(X/S)$ has a motive $\Sigma^\infty \mathrm{P}(X/S)_{\mathbb{Q}}$ in $\mathbf{DA}_1^{\mathrm{gsm}}(S)$.*

Proof. We have distinguished triangles

$$\Sigma^\infty(\underline{f}_* \mathbb{G}_{\mathrm{m}}) \otimes \mathbb{Q}) \to \Sigma^\infty P(X/S)_{\mathbb{Q}} \to \Sigma^\infty(\mathcal{P}ic^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q}) \xrightarrow{+}$$

and

$$\Sigma^\infty \mathcal{P}ic^{\mathrm{sm},\tau}_{X/S} \to \Sigma^\infty(\mathcal{P}ic^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q}) \to \Sigma^\infty \mathcal{NS}^{\mathrm{sm}}_{X/S} \otimes \mathbb{Q} \xrightarrow{+}.$$

The sheaf $\underline{f}_* \mathbb{G}_{\mathrm{m}} \simeq \mathrm{Res}_{\pi_0(f)} \mathbb{G}_{\mathrm{m}}$ is representable by a torus, the sheaf $\mathcal{P}ic^{\mathrm{sm},\tau}_{X/S}$ is representable by the abelian scheme $\mathrm{Pic}^{0,\mathrm{red}}_{X/S}$ because f is Pic-smooth (Theorem 3.2.2 (ii)), and the sheaf $\mathcal{NS}^{\mathrm{sm}}_{X/S}$ is representable by a lattice by Proposition 3.2.32. We conclude using Corollary 3.2.16. $\square$

Another important corollary is the comparison with the theory with transfers.

Corollary 3.2.35. *Let S be a regular scheme and $f : X \to S$ a smooth projective Pic-smooth morphism. The natural map*

$$a^{\mathrm{tr}} P(X/S) \longrightarrow P^{\mathrm{tr}}(X/S)$$

is an isomorphism.

Proof. By the same arguments as for $P(X/S)$, we have in this case distinguished triangles

$$(\underline{f}_* \mathbb{G}^{\mathrm{tr}}_{\mathrm{m}}) \otimes \mathbb{Q}) \to P(X/S)^{\mathrm{tr}}_{\mathbb{Q}} \to (\mathcal{P}ic^{\mathrm{sm},\mathrm{tr}}_{X/S} \otimes \mathbb{Q}) \xrightarrow{+}$$

and

$$\mathcal{P}ic^{\mathrm{sm},\tau,\mathrm{tr}}_{X/S} \to (\mathcal{P}ic^{\mathrm{sm},\mathrm{tr}}_{X/S} \otimes \mathbb{Q}) \to \mathcal{NS}^{\mathrm{sm},\mathrm{tr}}_{X/S} \otimes \mathbb{Q} \xrightarrow{+}.$$

with the analoguous notations, and each term of the triangles is represented by a smooth commutative group scheme. The result then follows from Proposition 2.2.10 $\square$

Finally, we look more closely at the case of a relative smooth projective curve, where things are simpler.

Proposition 3.2.36. *Let $f : C \to S$ be a smooth projective relative curve (S arbitrary). Then f is Pic-smooth, and $\mathcal{NS}_{C/S}$ is represented by a lattice canonically isomorphic to $\mathbb{Q}[\pi_0(C/S)]$. In particular, for any $g : T \to S$, the morphism $v_g : g^*P(C/S) \to P(C_T/T)$ is an isomorphism.*

Proof. When f has connected fibers, this is contained in [BLR90a, Theorem 9.3.1]. Since $\pi_0(C/S)$ is finite étale, the general case follows by étale descent. The addendum comes from the fact that the construction of $\pi_0(C/S)$ commutes with arbitrary base change. □

We also adopt a more traditional notation in this special case.

Notation 3.2.37. Let $f : C \to S$ be a smooth projective relative curve. We call the abelian scheme $\mathrm{Pic}^{0,\mathrm{red}}(C/S)$ the *(relative) Jacobian* of C over S, and we denote it by $\mathrm{Jac}(C/S)$.

Let $f : X \to S$ be a smooth projective Pic-smooth morphism of schemes. We introduce a morphism $\Theta_f : \Sigma^\infty \mathrm{P}(X/S) \longrightarrow f_*\mathbb{Q}_X(1)[2]$ which will be fundamental for the next section.

We start with the adjunction morphism

$$\mathrm{Sus}^1 \mathrm{Ev}_1 f_*\mathbb{Q}_X \xrightarrow{\eta} f_*\mathbb{Q}_X.$$

The functors Ev_1 and f_* commute, because they are right derived functors of right Quillen functors which commute at the model category level. We thus have a canonical isomorphism

$$f_* \mathrm{Ev}_1 \simeq \mathrm{Ev}_1 f_* : \mathbf{DA}^{\mathrm{eff}}(X) \longrightarrow \mathbf{DA}(S).$$

By composition we obtain a map

$$\mathrm{Sus}^1 f_* \mathrm{Ev}_1(\mathbb{Q}_X) \longrightarrow f_*\mathbb{Q}_X.$$

We then use the morphism w_S to obtain a map

$$\mathrm{Sus}^1 f_*(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1]) \longrightarrow f_*\mathbb{Q}_X.$$

Recall that $\mathrm{Sus}^1 \simeq \Sigma^\infty(-)(-1)[-2]$. By shifting and twisting, we get a morphism

$$\Sigma^\infty f_*(\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}[1]) \longrightarrow f_*\mathbb{Q}_X(1)[2].$$

Composing with the adjunction morphism $\tau_{\geq 0}(-) \to \mathrm{id}$ provides the desired morphism

$$\Theta_f : \Sigma^\infty \mathrm{P}(X/S) \longrightarrow f_*\mathbb{Q}_X(1)[2].$$

Remark 3.2.38. If S is regular, two steps of the construction above are isomorphisms by Proposition 3.2.13 and Lemma 3.2.22, which simplifies some later arguments.

We can do the same construction in $\mathbf{DM}(-)$ using w_S^{tr}, resulting in a morphism

$$\Theta_f^{\mathrm{tr}} : \Sigma^\infty_{\mathrm{tr}} \mathrm{P}^{\mathrm{tr}}(X/S) \longrightarrow f_*\mathbb{Q}_X(1)[2]$$

in $\mathbf{DM}(S)$. Modulo the isomorphism of Corollary 3.2.35 and the comparison theorem between $\mathbf{DA}$ and $\mathbf{DM}$, we can identify $a^{\mathrm{tr}}\Theta_f$ with Θ_f^{tr} for S normal.

Proposition 3.2.39. *Let $f : X \to S$ be a smooth projective Pic-smooth morphism of schemes. Let $g : T \to S$ be any morphism. Let $f' : X_T \to T$ be the pullback (which is still smooth projective Pic-smooth by Remark 3.2.26). The following diagram commutes in $\mathbf{DA}(S)$.*

$$\begin{tikzcd}
g^*\Sigma^\infty \mathrm{P}(X/S)_{\mathbb{Q}}(-1)[-2] \arrow[r, "\Theta_f"] \arrow[d, "v_g \circ R_g"'] & g^* f_*\mathbb{Q}_X \arrow[d, "\mathrm{Ex}^*_*"] \\
\Sigma^\infty \mathrm{P}(X_T/T)_{\mathbb{Q}}(-1)[-2] \arrow[r, "\Theta_{f'}"'] & f'_*\mathbb{Q}_{X_T}
\end{tikzcd}$$

Proof. This follows from a straighforward diagram composition argument using the commutative diagrams from Proposition 3.2.13, Lemma 3.2.24, and the naturality properties of the Ex^*_* morphisms (both derived and non-derived). We omit the details. □

3.3 Motivic Picard functor

We introduce and study the motivic Picard functor ω^1, which is a (mixed motivic, relative) generalisation of the Picard variety of a smooth projective variety over a field. We also study in parallel the 0-motivic analogue ω^0. Although some basic results on ω^0 from Section 3.3.1 are used in Sections 3.1 and 3.4, the main results are not used in the rest of the paper.

3.3.1 Definition and elementary properties

Definition 3.3.1. Let $n \geq 0$. The full embedding $\iota^n : \mathbf{DA}^n(S) \hookrightarrow \mathbf{DA}^{\mathrm{coh}}(S)$ preserves small sums, hence by Brown representability (see e.g. [Ayo07a, Proposition 2.1.21]) admits a right adjoint $\omega^n : \mathbf{DA}^{\mathrm{coh}}(S) \to \mathbf{DA}^n(S)$. We also write ω^n for the functor $\mathbf{DA}^{\mathrm{coh}}(S) \to \mathbf{DA}^{\mathrm{coh}}(S)$ obtained by postcomposing with ι^n. We write $\delta^n : \omega^n \to \mathrm{id}$ for the natural transformation induced by the counit.

Remark 3.3.2. The definition above can be extended to the whole of $\mathbf{DA}(S)$, but the resulting functors are not well-behaved; in particular, they do not respect compactness. Here is the simplest example of this phenomenon. Let k be an algebraically closed field. It is easy to see that the category $\mathbf{DA}_{0,c}(k)$ is equivalent to the bounded derived category of the category of finite dimensional $\mathbb{Q}$-vector spaces. In particular Hom groups in this category are finite dimensional. On the other hand, $\mathbf{DA}(k)(\mathbb{Q}_k, \mathbb{Q}_k(1)[1]) \simeq k^\times \otimes \mathbb{Q}$ (Proposition 3.B.4) is not finite dimensional in general. This shows $\omega^0(\mathbb{Q}(1))$ is not compact.

We start by giving some general formal properties of all the ω^n.

Proposition 3.3.3. *Let S be a noetherian finite-dimensional scheme.*

(i) *Let $M \in \mathbf{DA}^n(S)$. Then we have an isomorphism $\delta^n(M) : \omega^n(M) \simeq M$ and the natural transformation $\delta^n(\omega^n) : \omega^n \circ \omega^n \to \omega^n$ is invertible.*

(ii) *Let $f : T \to S$ be any morphism of schemes. There is a natural transformation $\alpha^n_f : f^*\omega^n \to \omega^n f^*$ making the triangles*

$$\begin{tikzcd} f^*\omega^n \arrow[r, "\alpha^n_f"] \arrow[dr, "f^*(\delta^n)"'] & \omega^n f^* \arrow[d, "\delta^n(f^*)"] \\ & f^* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \omega^n f^*\omega^n \arrow[r, "\delta^n(f^*\omega^n)"] \arrow[dr, "(\omega^n f^*)(\delta^n)"'] & f^*\omega^n \arrow[d, "\alpha^n_f"] \\ & \omega^n f^* \end{tikzcd}$$

commutative.

(iii) *Let $f : T \to S$ be any morphism of schemes. The natural transformation $\omega^n f_*(\delta^n)$ is invertible. Moreover there is a natural transformation $\beta^n_f : \omega^n f_* \to f_*\omega^n$ such that:*

a) *the following triangles*

$$\begin{tikzcd} \omega^n f_* \arrow[r, "\beta^n_f"] \arrow[dr, "\delta^n(f_*)"'] & f_*\omega^n \arrow[d, "f_*(\delta^n)"] \\ & f_* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \omega^n f_*\omega^n \arrow[r, "\omega^n(f_*\delta^n)"] \arrow[dr, "\delta^n(f_*\omega^n)"'] & \omega^n f_* \arrow[d, "\beta^n_f"] \\ & f_*\omega^n \end{tikzcd}$$

are commutative,

b) *$\omega^n(\beta^n_f)$ is invertible for any f, and*

c) *β^n_f is invertible for f finite.*

(iv) *Let $e : T \to S$ be a quasi-finite morphism of schemes. There exists a natural transformation $\eta^n_e : e_!\omega^n \to \omega^n e_!$ such that:*

a) *the following triangles*

$$\begin{tikzcd} e_!\omega^n \arrow[r, "\eta_e^n"] \arrow[dr, "e_!(\delta^n)"'] & \omega^n e_! \arrow[d, "\delta^n(e_!)"] \\ & e_! \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \omega^n e_! \omega^n \arrow[r, "\delta^n(e_!\omega^n)"] \arrow[dr, "(\omega^n e_!)(\delta^n)"'] & e_!\omega^n \arrow[d, "\eta_e^n"] \\ & \omega^n e_! \end{tikzcd}$$

commute, and

b) *when e is finite, η_e^n is invertible and coincides with β_e^{-1} modulo the natural isomorphism $e_! \simeq e_*$.*

(v) *Let $e : T \to S$ be a quasi-finite morphism. The natural transformation $\omega^n e^!(\delta^n)$ is invertible. Moreover there is a natural transformation $\gamma_e^n : \omega^n e^! \to e^!\omega^n$ such that:*

a) *the following triangles*

$$\begin{tikzcd} \omega^n e^! \arrow[r, "\gamma_e^n"] \arrow[dr, "\delta^n(e^!)"'] & e^!\omega^n \arrow[d, "e^!(\delta^n)"] \\ & e^! \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \omega^n e^! \omega^n \arrow[r, "\omega^n(e^!\delta^n)"] \arrow[dr, "\delta^n(e^!\omega^n)"'] & \omega^n e^! \arrow[d, "\gamma_e^n"] \\ & e^!\omega^n \end{tikzcd}$$

are commutative,

b) *$\omega^n(\gamma_e^n)$ is invertible for any e quasi-finite, and*

c) *γ_e^n is invertible for e étale.*

(vi) *Let $j : U \to S$ and $i : Z \to S$ be complementary open and closed immersions. Let $M \in \mathbf{DA}^{\mathrm{coh}}(S)$ with $j^*M \in \mathbf{DA}^n(S)$. Then the morphism $i^*\omega^n M \to \omega^n i^* M$ is invertible.*

Remark 3.3.4. The formulation of Proposition 3.3.3 follows closely the one of [AZ12, Proposition 2.16] about ω^0. More precisely, it is a direct generalization to all ω^n and to more general base schemes of all statements of loc. cit., minus the assertion in (ii) that α_f^0 is invertible for f smooth and the statement (vii) that ω^0 preserves compact objects. Unlike the others, these properties of ω^0 are not formal; in loc. cit., they follow from the key Proposition 2.11. We study their generalization to more general base schemes and higher n's below.

Proof. We can apply verbatim the proof of [AZ12, Proposition 2.16] up to the sentence "To complete the proof (...)" on page 319. Notice that the rest of the proof after that sentence establishes the last assertion in (ii) together with (vii), which are precisely the points we are not claiming.

More precisely, up to that sentence, the proof of loc. cit. uses only general properties of $\mathbf{DA}$, the definition of ω^0 as adjoint, and the following permanence properties of cohomological 0-motives under the six operations.

- For all morphisms f, the functor f^* preserves $\mathbf{DA}^0$.
- For all finite morphisms f, the functor f_* preserves $\mathbf{DA}^0$.
- For all quasi-finite morphism e, the functor $e_!$ preserves $\mathbf{DA}^0$.

The generalisation of these properties to $\mathbf{DA}^n$ are established in the necessary generality in Proposition 3.1.17. ☐

Proposition 3.3.5. *Let S be noetherian of finite dimension.*

(i) *Let $f : X \to S$ be a smooth proper morphism of schemes. Let $X \xrightarrow{f^\circ} \pi_0(X/S) \xrightarrow{\pi_0(f)} S$ be its Stein factorisation, with $\pi_0(f)$ automatically finite étale. Then there is a canonical isomorphism*

$$\omega^0(f_*\mathbb{Q}_X) \xrightarrow{\sim} \pi_0(f)_*\mathbb{Q}_{\pi_0(X/S)}.$$

(ii) *The functor ω^0 preserves geometrically smooth objects. More precisely, it sends $\mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm}}(S)$ to $\mathbf{DA}^{0}_{\mathrm{gsm}}(S)$ and $\mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm},c}(S)$ to $\mathbf{DA}^{0}_{\mathrm{gsm},c}(S)$. Moreover, for any $M \in \mathbf{DA}^{\mathrm{coh}}_{\mathrm{gsm}}(S)$ and any morphism $f : T \to S$, the natural morphism $\alpha^0_f(M) : f^*\omega^0 M \to \omega^0 f^* M$ is an isomorphism.*

(iii) *The morphism α^0_f is invertible for f smooth.*

(iv) *The functor ω^0 preserves compact objects. More precisely, it sends $\mathbf{DA}^{\mathrm{coh}}_{c}(S)$ to $\mathbf{DA}^{0}_{c}(S)$.*

Remark 3.3.6. These results were proved in [AZ12, §2] under the assumption that S is quasi-projective over a field k and f is projective.

Proof. It is easy to see from the definition of geometrically smooth motives and the fact that π_0 commutes with base change that point (ii) follows from (i). We now notice that the end of the proof of [AZ12, Proposition 2.16] (starting at “To complete the proof (...)”), which deduces (iii) and (iv) in the situation of loc. cit. from [AZ12, Proposition 2.11], applies verbatim and reduce Statements (ii)-(iv) to the sole Statement (i).

To prove Statement (i), it is enough by the Yoneda lemma to establish that for all $L \in \mathbf{DA}^0(S)$, the natural map $\pi_0(f)_*\mathbb{Q} \to f_*\mathbb{Q}_X$ induces an isomorphism

$$\mathbf{DA}(S)(L, \pi_0(f)_*\mathbb{Q}) \xrightarrow{\sim} \mathbf{DA}(S)(L, f_*\mathbb{Q}_X).$$

By Proposition 3.1.28, we have $\mathbf{DA}^0(S) = \mathbf{DA}_0(S)$. It is thus enough to show that for all $e : U \to S$ étale and $n \in \mathbb{Z}$, we have an isomorphism

$$\mathbf{DA}(S)(e_\sharp\mathbb{Q}_U[-n], \pi_0(f)_*\mathbb{Q}) \xrightarrow{\sim} \mathbf{DA}(S)(e_\sharp\mathbb{Q}_U[-n], f_*\mathbb{Q}_X).$$

By adjunction, proper base change, and the fact that π_0 commutes with base change, we see that we can assume $e = \mathrm{id}$. We are thus left to prove that for all $n \in \mathbb{Z}$, we have

$$\mathbf{DA}(\pi_0(X/S))(\mathbb{Q}, \mathbb{Q}[n]) \xrightarrow{\sim} \mathbf{DA}(X)(\mathbb{Q}, \mathbb{Q}[n])$$

where the morphism is induced by pullback by f°. The morphism f° is smooth proper with geometrically connected fibers, so this follows from Proposition 3.B.3 (iv). $\square$

Here are some interesting corollaries of Proposition 3.3.5.

Corollary 3.3.7. *Let S be a noetherian finite-dimensional scheme.*

(i) *Let M be in $\mathbf{DA}_{\mathrm{hom}}(S)$ and N be in $\mathbf{DA}^{\mathrm{coh}}(S)$. Then the morphism $\delta^0(N)$ induces an isomorphism*

$$\mathbf{DA}(S)(M, \omega^0 N) \underset{\sim}{\xrightarrow{\delta^0(N)_*}} \mathbf{DA}(S)(M, N).$$

(ii) *We have $\mathbf{DA}_{\mathrm{hom}}(S) \cap \mathbf{DA}^{\mathrm{coh}}(S) = \mathbf{DA}^0(S)$.*

(iii) *For all $N \in \mathbf{DA}^{\mathrm{coh}}(S)$ we have $\omega^0(N(-1)) \simeq 0$.*

(iv) *For all $N \in \mathbf{DA}^{\mathrm{coh}}(S)$ and $d \geq 1$, we have*

$$\omega^1(N(-d)) \simeq \begin{cases} (\omega^0 N)(-1), & d = 1 \\ 0, & d \geq 2 \end{cases}.$$

Proof. We first prove (i). It is enough to show the isomorphism for the generators $M = g_\sharp\mathbb{Q}_X[n]$ for $g : X \to S$ a smooth morphism and $n \in \mathbb{Z}$. By naturality of the adjunction which underlies δ^0, we have a commutative square

$$\begin{tikzcd}
\mathbf{DA}(S)(g_\sharp\mathbb{Q}_X[n], \omega^0 N) \arrow[r, "\delta^0(N)_*"] \arrow[d, "\sim"'] & \mathbf{DA}(S)(g_\sharp\mathbb{Q}_X[n], N) \arrow[d, "\sim"] \\
\mathbf{DA}(X)(\mathbb{Q}_X[n], g^*\omega^0 N) \arrow[r, "\delta^0(N)_*"] & \mathbf{DA}(X)(\mathbb{Q}_X[n], g^* N).
\end{tikzcd}$$

The first commutative triangle in Proposition 3.3.3 (ii) shows that we have a commutative square

$$\begin{tikzcd}
\mathbf{DA}(X)(\mathbb{Q}_X[n], g^*\omega^0 N) \arrow[r, "\delta^0(N)_*"] \arrow[d, "\alpha_g(N)"'] & \mathbf{DA}(X)(\mathbb{Q}_X[n], g^*N) \arrow[d, equal] \\
\mathbf{DA}(X)(\mathbb{Q}_X[n], \omega^0 g^*N) \arrow[r, "\delta^0(g^*N)_*"'] & \mathbf{DA}(X)(\mathbb{Q}_X[n], g^*N).
\end{tikzcd}$$

The left vertical map is an isomorphism by Proposition 3.3.5 (ii), and the bottom map is an isomorphism because $\mathbb{Q}_X[n]$ is a cohomological 0-motive. Putting this together with the previous commutative square concludes the proof of (i).

Statement (ii) follows directly from (i) applied to the identity map of an object in $\mathbf{DA}^{\mathrm{coh}}(S) \cap \mathbf{DA}_{\mathrm{hom}}(S)$.

To prove Statement (iii), we must show that for all $L \in \mathbf{DA}^0(S)$, we have $\mathbf{DA}(S)(L, N(-1)) = 0$. Since $\mathbf{DA}^0(S) = \mathbf{DA}_0(S)$ by Proposition 3.1.28 and $\mathbf{DA}_{\mathrm{hom}}(S)$ is stable by positive twists by Proposition 3.1.10 (iv), the motive $L(1)$ is homological. By (i), this implies that $\mathbf{DA}(S)(L(1), N) \simeq \mathbf{DA}(S)(L(1), \omega^0 N)$. In other words, we can assume that both L and N are 0-motives. The statement to be proven is triangulated and commutes with infinite sums in L, so that we can assume that L is a generator of the form $e_\sharp \mathbb{Q}_U[n]$ for $e : U \to S$ an étale morphism and $n \in \mathbb{Z}$. Since this is a compact object, we can similarly assume that N is a generator of $\mathbf{DA}^0(S)$, of the form $f_*\mathbb{Q}_V[m]$ for $f : V \to S$ a finite morphism. We then have $\mathbf{DA}(S)(L, N(-1)) \simeq \mathbf{DA}(U \times_S V)(\mathbb{Q}, \mathbb{Q}(-1)[m-n])$. This group vanishes by Proposition 3.B.2.

By (iii), we only need to establish (iv) in the case $d = 1$. The motive $\omega^0(N)(-1)$ is in $\mathbf{DA}^1(S)$ by Proposition 3.1.10 (ii). Hence by the Yoneda lemma, it is enough to show that for all $M \in \mathbf{DA}^1(S)$, the map $\delta^0(N)$ induces an isomorphism

$$\mathbf{DA}(S)(M, (\omega^0 N)(-1)) \xrightarrow[\sim]{\delta^0(N)_*} \mathbf{DA}(S)(M, N(-1)).$$

By Proposition 3.1.28, we have $\mathbf{DA}^1(S) = \mathbf{DA}_1(S)(-1)$. Write $M = M'(-1)$ with $M' \in \mathbf{DA}_1(S)$. In particular, M' is an homological motive. We have a commutative square

$$\begin{tikzcd}
\mathbf{DA}(S)(M, (\omega^0 N)(-1)) \arrow[r, "\delta^0(N)_*"] \arrow[d, "\sim"'] & \mathbf{DA}(S)(M, N(-1)) \arrow[d, "\sim"] \\
\mathbf{DA}(S)(M', \omega^0 N) \arrow[r, "\delta^0(N)_*"'] & \mathbf{DA}(S)(M', N)
\end{tikzcd}$$

The bottom map is an isomorphism by (i), and this concludes the proof in case $d = 1$. $\square$

We now compute ω^0 for some motives attached to commutative group schemes.

Proposition 3.3.8. (i) *Let G be an abelian scheme or a lattice over S; then $\omega^0(\Sigma^\infty G_{\mathbb{Q}}(-1)) \simeq 0$.*

(ii) *Assume S is normal. Let T be a torus over S. Let $X_*(T)$ be the cocharacter lattice of T. Then*
$\omega^0(\Sigma^\infty T_{\mathbb{Q}}(-1)) \simeq \Sigma^\infty X_*(T)_{\mathbb{Q}}$.

(iii) *Assume S is normal. Let $\mathbb{M} \in \mathcal{M}_1(S)$ and $W_{-2}\mathbb{M}$ be its toric part. Then $\omega^0(\mathcal{R}(\mathbb{M})(-1)) \simeq \Sigma^\infty X_*(W_{-2}\mathbb{M})_{\mathbb{Q}}$.*

Proof. First of all, we note that the objects to which we wish to apply ω^0 are in $\mathbf{DA}^1(S) \subset \mathbf{DA}^{\mathrm{coh}}(S)$ by Corollary 3.2.16 and Proposition 3.1.28.

We first prove (i). Let G be an abelian scheme or a lattice and $M = \Sigma^\infty G_{\mathbb{Q}}$. The category $\mathbf{DA}^0(S) = \mathbf{DA}_0(S)$ is compactly generated by objects of the form $e_\sharp \mathbb{Q}_U$ for $e : U \to S$ étale. By adjunction and Proposition 3.2.4, we reduce to showing that for all $n \in \mathbb{Z}$, we have

$\mathbf{DA}(S)(\mathbb{Q}_S, M[n]) = 0$. By using an h-hypercovering of S with regular terms, cohomological h-descent for $\mathbf{DA}(-)$ [CDb, Theorem 14.3.4] and Proposition 3.2.4, we reduce to the case where S is regular.

If G is a lattice, we can then write its motive as a direct factor $f_*\mathbb{Q}$ for f finite étale, and we are done by adjunction and Proposition 3.B.2. If G is an abelian scheme, we know from [AHPL14, Theorem 3.3] (essentially, in this case, the theorem of Deninger and Murre) that the motive $\Sigma^\infty G_\mathbb{Q}$ is geometrically smooth, thus smooth, and compact. We reduce to the case where S is the spectrum of a field by combining colocalisation, absolute purity in the form of Proposition 3.1.7 and continuity with the vanishing statement of Corollary 3.3.7 (iii). When S is the spectrum of a field, we can write G as direct factor of the Jacobian of a smooth projective geometrically connected curve $f : C \to \mathbf{Spec}(k)$ with a rational point [Kat99, Theorem 11]. By Proposition 3.2.8 and relative purity, we have

$$\mathbb{Q}(-1)[-2] \oplus \Sigma^\infty \operatorname{Jac}(C)_\mathbb{Q}(-1)[-2] \oplus \mathbb{Q} \simeq f_*\mathbb{Q}_C.$$

We have $\mathbf{DA}(k)(\mathbb{Q}_k, \mathbb{Q}_k(-1)[n]) = 0$ for all n (Proposition 3.B.2). By adjunction, we have $\mathbf{DA}(k)(\mathbb{Q}_k, f_*\mathbb{Q}_C[n]) \simeq \mathbf{DA}(C)(\mathbb{Q}_C, \mathbb{Q}_C[n])$ which is isomorphic to $\mathbb{Q}$ for $n = 0$ and 0 otherwise (Proposition 3.B.3). Similarly, we have $\mathbf{DA}(k)(\mathbb{Q}_k, \mathbb{Q}_k[n])$ is isomorphic to $\mathbb{Q}$ for $n = 0$ and 0 otherwise. Putting everything together, for any n we deduce that $\mathbf{DA}(k)(\mathbb{Q}_k, \Sigma^\infty \operatorname{Jac}(C)_k[n]) = 0$, as required.

We prove (ii). Let T be a torus. We have $\Sigma^\infty T_\mathbb{Q}(-1) \simeq \Sigma^\infty X_*(T)_\mathbb{Q}$ by Corollary 3.2.11 (using the assumption that S is normal). The motive $\Sigma^\infty X_*(T)_\mathbb{Q}$ lies in $\mathbf{DA}_0(S)$: this can be tested pointwise by Proposition 3.1.24, and over a field a lattice is a direct factor of the motive of a finite étale morphism. This concludes the proof.

Finally, (iii) follows immediately from the two previous points. □

Corollary 3.3.9. *Assume S regular. Let $f : X \to S$ be a smooth projective Pic-smooth morphism of schemes. Then there is an isomorphism*

$$\omega_0(\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[-2]) \simeq \pi_0(f)_*\mathbb{Q}$$

Proof. First, by Corollary 3.2.34, Proposition 3.2.14 and Proposition 3.1.28, the motive $\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[-2]$ is in $\mathbf{DA}^1(S)$, and it makes sense to apply ω^0. More precisely, the devissage in the proof of Corollary 3.2.34 together with Proposition 3.3.8 shows that there is an isomorphism

$$\omega_0(\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[-2]) \simeq \Sigma^\infty X_*(\operatorname{Res}_{\pi_0(f)} \mathbb{G}_\mathrm{m})_\mathbb{Q}.$$

The cocharacter lattice of the Weil restriction $\operatorname{Res}_{\pi_0(f)} \mathbb{G}_\mathrm{m}$ is the permutation lattice associated to $\pi_0(f)$; hence, $\Sigma^\infty X_*(\operatorname{Res}_{\pi_0(f)} \mathbb{G}_\mathrm{m})_\mathbb{Q} \simeq \pi_0(f)_*\mathbb{Q}$ as required. □

Finally, here is a negative result which shows that the ω^n's are not well-behaved, at least over "large" fields.

Proposition 3.3.10. *Let $n \geq 2$ and k be an algebraically closed field of infinite transcendence degree over $\mathbb{Q}$, e.g. $k = \mathbb{C}$. Then $\omega^n : \mathbf{DA}^{\mathrm{coh}}(k) \to \mathbf{DA}^n(k)$ does not preserve compact objects.*

Proof. We prove this by contradiction. Assume that ω^n preserves compact object and write again $\omega^n : \mathbf{DA}_c^{\mathrm{coh}}(k) \to \mathbf{DA}_c^n(k)$ for the restriction. By Proposition 3.1.26, the duality functor $\mathbb{D}_k$ restricts to anti-equivalences of categories $\mathbf{DA}_c^{\mathrm{coh}}(k)^{\mathrm{op}} \simeq \mathbf{DA}_{\mathrm{hom},c}(k)$ and $\mathbf{DA}_c^n(k)^{\mathrm{op}} \simeq \mathbf{DA}_{n,c}(k)$. This implies that the composition $\mathbb{D}_k \circ (\omega^n)^{\mathrm{op}} \circ \mathbb{D}_k : \mathbf{DA}_{\mathrm{hom},c}(k) \to \mathbf{DA}_{n,c}(k)$ provides a left adjoint to the inclusion $\mathbf{DA}_{n,c}(k) \to \mathbf{DA}_{\mathrm{hom},c}(k)$.

By the equivalence between $\mathbf{DA}$ and $\mathbf{DM}$ [CDb, Corollary 16.2.22] and cancellation [Voe10], this also provides a left adjoint to $\mathbf{DM}_{n,c}^{\mathrm{eff}}(k) \to \mathbf{DM}_c^{\mathrm{eff}}(k)$. This contradicts [ABV09, §2.5] (note that the assumption there is the existence of a left adjoint to $\mathbf{DM}_n^{\mathrm{eff}}(k) \to \mathbf{DM}^{\mathrm{eff}}(k)$ but the proof only uses the existence of the adjoint on compact objects). □

3.3.2 Computation for smooth proper families

We can now compute ω^1 in an important special case.

Theorem 3.3.1. *Let $f : X \to S$ be a smooth projective Pic-smooth morphism with S regular. The morphism $\Theta_f : (\Sigma^\infty \mathrm{P}(X/S)_{\mathbb{Q}})(-1)[-2] \to f_*\mathbb{Q}_X$ of Section 3.2.3 induces an isomorphism*

$$\omega^1 f_*\mathbb{Q}_X \simeq (\Sigma^\infty \mathrm{P}(X/S)_{\mathbb{Q}})(-1)[-2].$$

In particular, under these hypotheses, $\omega^1 f_*\mathbb{Q}_X$ is compact.

Proof. First of all, the motive $\Sigma^\infty(\mathrm{P}(X/S)_{\mathbb{Q}})$ lies in $\mathbf{DA}_1(S)$ by Corollary 3.2.34. By Proposition 3.1.28, this implies that $\Sigma^\infty(\mathrm{P}(X/S)\otimes\mathbb{Q})(-1)[-2]$ lies in $\mathbf{DA}^1(S)$. This implies that Θ_f induces a morphism $\Sigma^\infty(\mathrm{P}(X/S)\otimes\mathbb{Q})(-1)[-2] \to \omega^1 f_*\mathbb{Q}_X$; the claim is that this is an isomorphism.

We first treat the case when S is the spectrum of a field k. Let k^{perf} be a perfect closure of k and $h : \mathbf{Spec}(k^{\mathrm{perf}}) \to \mathbf{Spec}(k)$ be the canonical morphism. By Proposition 3.2.39 and applying ω^1, we have a commutative diagram

$$\begin{tikzcd}
h^*\Sigma^\infty \mathrm{P}(X/S)_{\mathbb{Q}}(-1)[-2] \arrow[r] \arrow[d, "v_h\circ R_h"'] & \omega^1(h^*f_*\mathbb{Q}_X) \arrow[d, "\omega^1(\mathrm{Ex}^*_*)"] \\
\Sigma^\infty \mathrm{P}(X_T/T)_{\mathbb{Q}}(-1)[-2] \arrow[r, "\Theta_{f'}"'] & \omega^1(f'_*\mathbb{Q}_{X_T}).
\end{tikzcd}$$

The morphism h is not smooth and we cannot directly apply Lemma 3.2.24. However, since f is Pic-smooth, the morphism $v_h : h^*P(X/k) \to P(X_{k^{\mathrm{perf}}}/k^{\mathrm{perf}})$ is an isomorphism if and only if the natural morphism $h^*\mathcal{NS}(X/k) \to \mathcal{NS}(X_{k^{\mathrm{perf}}}/k^{\mathrm{perf}})$ is. Let k^s be a separable closure of k and $\bar{k} = k^s k^{\mathrm{perf}}$. Looking at the proof of Proposition 3.2.32, we find that $\mathcal{NS}(X/k)$ is represented by the $\mathrm{Gal}(k^s/k)$-module $\mathrm{NS}(X_{k^s}/k^s)$ while $\mathcal{NS}(X_{k^{\mathrm{perf}}}/k^{\mathrm{perf}})$ is represented by the $\mathrm{Gal}(\bar{k}/k^{\mathrm{perf}})$-module $\mathrm{NS}(X_{\bar{k}}/\bar{k})$. Those two groups are canonically isomorphic, and we conclude that v_h is an isomorphism. Since R_h is an isomorphism, we see that the left vertical map in the diagram is an isomorphism. Moreover, since h is finite and purely inseparable, by the separation property of $\mathbf{DA}$ and Lemma 3.1.19 (ii), we see that the natural morphism $\alpha_h : h^*\omega^1 \to \omega^1 f^*$ is an isomorphism. Together with the commutative diagram above, this shows that we can reduce the question of whether Θ_f is an isomorphism to the case of a perfect field.

Let us assume that k is perfect. By Proposition 3.1.28 and Proposition 3.1.26, the category $\mathbf{DA}^1(k)$ is compactly generated by motives of the form $g_\sharp\mathbb{Q}_C(-1)$ for a smooth projective curve $g : C \to k$. We thus have to show that for all such g and all $n \in \mathbb{Z}$, the map

$$\mathbf{DA}(k)(g_\sharp\mathbb{Q}_C(-1)[-n], (\Sigma^\infty \mathrm{P}(X/k)_{\mathbb{Q}})(-1)[-2]) \xrightarrow{\Theta_{f*}} \mathbf{DA}(k)(g_\sharp\mathbb{Q}_C(-1)[-n], f_*\mathbb{Q}_X)$$

induced by Θ_f is an isomorphism.

Let us look at the left hand side. We have a sequence of isomorphisms

$$\begin{aligned}
\mathbf{DA}(g_\sharp\mathbb{Q}_C, \Sigma^\infty P(X/k)[n-2]) &\simeq \mathbf{DM}(a^{\mathrm{tr}}g_\sharp\mathbb{Q}_C, a^{\mathrm{tr}}\Sigma^\infty P(X/k)[n-2]) \\
&\simeq \mathbf{DM}(M_k^{\mathrm{tr}}(C), \Sigma^\infty a^{\mathrm{tr}}P(X/k)[n-2]) \\
&\simeq \mathbf{DM}^{\mathrm{eff}}(M_k^{\mathrm{eff,tr}}(C), a^{\mathrm{tr}}P(X/k)[n-2]) \\
&\simeq \mathbf{DM}^{\mathrm{eff}}(M_k^{\mathrm{eff,tr}}(C), P^{\mathrm{tr}}(X/k)[n-2])
\end{aligned}$$

where the first line comes from the $\mathbf{DA}\,/\,\mathbf{DM}$ comparison result [CDb, Corollary 16.2.22], the second line follows from Lemma 1.2.5, the third line follows from the cancellation theorem [Voe10] and the last line comes from Corollary 3.2.35. The object $P^{\mathrm{tr}}(X/k)$ fits into the following triangles in $\mathbf{DM}^{\mathrm{eff}}(k)$

$$\underline{f}_*\mathbb{G}_{\mathrm{m}}^{\mathrm{tr}}\otimes\mathbb{Q} \to P^{\mathrm{tr}}(X/k)_{\mathbb{Q}} \to \mathcal{P}ic^{\mathrm{sm,tr}}_{X/k}\otimes\mathbb{Q} \xrightarrow{+}$$

and

$$\mathcal{P}ic^{\mathrm{sm},\tau,\mathrm{tr}}_{X/k}\otimes\mathbb{Q} \to \mathcal{P}ic^{\mathrm{sm,tr}}_{X/k}\otimes\mathbb{Q} \to \mathcal{NS}^{\mathrm{sm,tr}}_{X/k}\otimes\mathbb{Q} \xrightarrow{+}.$$

We deduce the existence of long exact sequences

$$
\begin{tikzcd}
\dots \arrow[r] & \mathbf{DM}^{\mathrm{eff}}(k)(M^{\mathrm{eff,tr}}(C), \underline{f}_*\mathbb{G}_{\mathrm{m}}^{\mathrm{tr}}[n]) \arrow[r] & \mathbf{DM}^{\mathrm{eff}}(k)(M^{\mathrm{eff,tr}}(C), P^{\mathrm{tr}}(X/k)_{\mathbb{Q}}[n]) \arrow[d] \\
& \dots & \mathbf{DM}^{\mathrm{eff}}(k)(M^{\mathrm{eff,tr}}(C), \mathcal{P}ic^{\mathrm{sm,tr}}_{X/k} \otimes \mathbb{Q}[n]) \arrow[l]
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\dots \arrow[r] & \mathbf{DM}^{\mathrm{eff}}(k)(M^{\mathrm{eff,tr}}(C), \mathcal{P}ic^{\mathrm{sm},\tau,\mathrm{tr}}_{X/k}) \arrow[r] & \mathbf{DM}^{\mathrm{eff}}(k)(M^{\mathrm{eff,tr}}(C), \mathcal{P}ic^{\mathrm{sm,tr}}_{X/k} \otimes \mathbb{Q}) \arrow[d] \\
& \dots & \mathbf{DM}^{\mathrm{eff}}(k)(M^{\mathrm{eff,tr}}(C), \mathcal{NS}^{\mathrm{sm,tr}}_{X/k} \otimes \mathbb{Q}). \arrow[l]
\end{tikzcd}
$$

The sheaves with transfers occuring in those terms are representable by commutative group schemes (a torus, an abelian variety, a lattice) which are homotopy invariant; hence, by [MVW06, Corollary 14.9], we can compute these terms as Zariski cohomology groups of sheaves on C. The Zariski cohomology of a torus on a curve involves $\mathcal{O}^\times$ and Pic, while abelian varieties and lattices are flasque for the Zariski topology. After a bit of work, this shows that

$$\mathbf{DM}^{\mathrm{eff}}(M_k^{\mathrm{eff,tr}}(C), P^{\mathrm{tr}}(X/k)[n-2]) \simeq \begin{cases} \mathcal{O}^\times(X \times_k C) \otimes \mathbb{Q}, \ n = 1 \\ \mathrm{Pic}(X \times_k C) \otimes \mathbb{Q}, \ n = 2 \\ 0, \ n \neq 1, 2 \end{cases}$$

On the other hand, we have

$$\mathbf{DA}(k)(g_\sharp \mathbb{Q}_C(-1)[-n], f_*\mathbb{Q}_X) \simeq \mathbf{DA}(X \times_k C)(\mathbb{Q}, \mathbb{Q}(1)[n])$$

by adjunction and proper base change. By Proposition 3.B.4, this motivic cohomology is

$$\mathbf{DA}(k)(g_\sharp \mathbb{Q}_C(-1)[-n], f_*\mathbb{Q}_X) \simeq \begin{cases} \mathcal{O}^\times(X \times_k C) \otimes \mathbb{Q}, \ n = 1 \\ \mathrm{Pic}(X \times_k C) \otimes \mathbb{Q}, \ n = 2 \\ 0, \ n \neq 1, 2 \end{cases}$$

Both sides are obtained by comparison with sheaf cohomology of $\mathbb{G}_{\mathrm{m}}$, and a careful analysis of the construction of Θ_f and $\nu^{n,1}$ shows that

$$\mathbf{DA}(k)(g_\sharp \mathbb{Q}_C(-1)[-n], (\Sigma^\infty \mathrm{P}(X/k)_{\mathbb{Q}})(-1)[-2]) \xrightarrow{\Theta_{f*}} \mathbf{DA}(k)(g_\sharp \mathbb{Q}_C(-1)[-n], f_*\mathbb{Q}_X)$$

is an isomorphism, as needed.

We now do the general case. We can assume S is connected, and so integral. The statement of the theorem is equivalent to the following: for all $M \in \mathbf{DA}_1(S)$, the map Θ_f induces an isomorphism

$$\mathbf{DA}(S)(M, (\Sigma^\infty \mathrm{P}(X/S)_{\mathbb{Q}})(-1)[-2]) \xrightarrow{\sim} \mathbf{DA}(S)(M, f_*\mathbb{Q}_X).$$

We first make a series of reformulations of this statement. By Proposition 3.1.28 and the definition of $\mathbf{DA}_1(S)$, the category $\mathbf{DA}^1(S)$ is compactly generated by objects of the form $g_\sharp \mathbb{Q}_C(-1)$ for a smooth curve $g : C \to S$. We can thus reformulate the theorem as follows: for every smooth curve $g : C \to S$ and all $n \in \mathbb{Z}$, the map

$$\mathbf{DA}(S)(g_\sharp \mathbb{Q}_C(-1)[-n], (\Sigma^\infty \mathrm{P}(X/S)_{\mathbb{Q}})(-1)[-2]) \xrightarrow{\Theta_{f*}} \mathbf{DA}(S)(g_\sharp \mathbb{Q}_C(-1)[-n], f_*\mathbb{Q}_X)$$

induced by Θ_f is an isomorphism. By adjunction, this is equivalently to the statement that the map

$$\mathbf{DA}(C)(\mathbb{Q}_C(-1)[-n], g^*(\Sigma^\infty \mathrm{P}(X/S)_{\mathbb{Q}})(-1)[-2]) \xrightarrow{(g^*\Theta_f)_*} \mathbf{DA}(C)(\mathbb{Q}_C(-1)[-n], g^* f_*\mathbb{Q}_X)$$

induced by $g^*\Theta_f$ is an isomorphism. Let $f' : X_C \to C$ be the pullback of f along g. Since g is smooth, we can apply the commutative diagram of Proposition 3.2.39 which shows that $(g^*\Theta_f)_*$ above is an isomorphism if and only the morphism

$$\mathbf{DA}(C)(\mathbb{Q}_C, (\Sigma^\infty \mathrm{P}(X_C/C)_\mathbb{Q})[n-2]) \xrightarrow{\Theta_{f'}} \mathbf{DA}(C)(\mathbb{Q}_C, f'_*\mathbb{Q}_{X_C}(1)[n])$$

is an isomorphism. In other words, since f' still satisfies all the hypotheses of the theorem, we can assume that $g = \mathrm{id}$.

By adjunction, the right-hand side is isomorphic to the motivic cohomology group $H^{n,1}_{\mathcal{M}}(X)$. Because S is regular, we know from Proposition 3.B.4 how to compute it: it is zero for $n \neq 1, 2$, and we have explicit morphisms relating it to $\mathcal{O}^\times(X)_\mathbb{Q}$ if $n = 1$ (resp. $\mathrm{Pic}(X)_\mathbb{Q}$ if $n = 2$). The idea of the rest of the proof is to apply a similar localisation argument to the proof of Proposition 3.B.4 to the group

$$\mathrm{HP}^{n-2}(X/S) := \mathbf{DA}(S)(\mathbb{Q}_S, (\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q})[n-2]).$$

Let $j : U \to S$ be a non-empty open set and $i : Z \to S$ its reduced closed complement. Then by applying colocalisation, we get a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathbf{DA}(Z)(\mathbb{Q}_Z, i^!\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}[n-2]) \arrow[r] \arrow[d] & \mathrm{HP}^{n-2}(X/S) \arrow[r] \arrow[d] & \mathrm{HP}^{n-2}(X_U/U) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & \mathbf{DA}(Z)(\mathbb{Q}_Z, i^!(f_*\mathbb{Q}_X(1)[n])) \arrow[r] & H^{n,1}_{\mathcal{M}}(X) \arrow[r] & H^{n,1}_{\mathcal{M}}(X_U) \arrow[r] & \ldots
\end{tikzcd}$$

As in the proof of Proposition 3.B.4, we stratify $Z = Z_0 \subset Z_1 \subset \ldots \subset Z_d = \emptyset$ in such a way that for all k, the scheme $(Z_k \setminus Z_{k+1})_{\mathrm{red}}$ is regular of codimension d_k in S and in such a way that $(Z \setminus Z_1)$ contains all points of codimension 1 of Z in S (so that $d_k \leq 2$ for $k \geq 1$). Let $i_k : (Z_k \setminus Z_{k+1})_{\mathrm{red}} \to S$ be the corresponding regular locally closed immersion.

By Corollary 3.2.34, the motive $\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)$ is in $\mathbf{DA}^1_{\mathrm{gsm}}(S)$. By Proposition 3.1.7, for any k, we have $i_k^!\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q} \simeq i^*\mathrm{P}(X/S)_\mathbb{Q}(-d_k)[-2d_k]$. In particular, by Corollary 3.3.7 (iii), we have $\omega^0(i_k^!\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}) \simeq 0$ for $k \geq 2$. This shows that by applying inductively absolute purity and colocalisation, we get a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathbf{DA}(Z)(\mathbb{Q}_{Z\setminus Z_1}, i_1^*\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[n-4]) \arrow[r] \arrow[d] & \mathrm{HP}^{n-2}(X/S) \arrow[r] \arrow[d] & \mathrm{HP}^{n-2}(X_U/U) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & H^{n-2,0}(X_{Z\setminus Z_1}) \arrow[r] & H^{n,1}_{\mathcal{M}}(X) \arrow[r] & H^{n,1}_{\mathcal{M}}(X_U) \arrow[r] & \ldots
\end{tikzcd}$$

As in the proof of Proposition 3.B.4, we make the notational abuse of replacing $Z \setminus Z_1$ by Z in the rest of the proof, since everything happens in codimension 1. The motive $i^*\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[n-4]$ lies in $\mathbf{DA}^{\mathrm{coh}}(Z)$, so that

$$\mathbf{DA}(Z)(\mathbb{Q}_Z, i^*\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[n-4]) \simeq \mathbf{DA}(Z)(\mathbb{Q}_Z, \omega^0(i^*\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[n-4]))$$

Using Corollary 3.2.34, we apply Proposition 3.3.5 (ii) to get an isomorphism

$$\omega^0(i^*\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[n-4]) \simeq i^*\omega^0(\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[n-4]).$$

By Corollary 3.3.9, we then have

$$\omega^0(\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}(-1)[n-4]) \simeq \pi_0(f)_*\mathbb{Q}[n-2].$$

We deduce that

$$\mathbf{DA}(Z)(\mathbb{Q}_Z, i^!\Sigma^\infty \mathrm{P}(X/S)_\mathbb{Q}[n-2]) \simeq H^{n-2,0}(\pi_0(X_Z/Z))$$

We rewrite this into the previous commutative diagram to get

$$\begin{tikzcd}
\ldots \arrow[r] & H^{n-2,0}(\pi_0(X_Z/Z)) \arrow[r] \arrow[d, "(\pi_0)^*"'] & \mathrm{HP}^{n-2}(X/S) \arrow[r] \arrow[d] & \mathrm{HP}^{n-2}(X_U/U) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & H^{n-2,0}(X_Z) \arrow[r] & H^{n,1}_{\mathcal{M}}(X) \arrow[r] & H^{n,1}_{\mathcal{M}}(X_U) \arrow[r] & \ldots
\end{tikzcd}$$

By Proposition 3.B.3, since X_Z and $\pi_0(X_Z/Z)$ are both regular and have the same set of connected components, the map $(\pi_0)^*$ is an isomorphism for all n, and the groups $H^{n-2,0}(X_Z)$ vanish for $n \neq 2$. As a consequence, we see that the pullback map $\mathrm{HP}^{n-2}(S) \to \mathrm{HP}^{n-2}(U)$ is an isomorphism for $n \neq 1, 2$, and there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{HP}^{-1}(X/S) \arrow[r] \arrow[d] & \mathrm{HP}^{-1}(X_U/U) \arrow[r] \arrow[d] & \mathbb{Q}^{\pi_0(X_Z)} \arrow[r] \arrow[d, equal] & \mathrm{HP}^0(X/S) \arrow[r] \arrow[d] & \mathrm{HP}^0(X_U/U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{1,1}_{\mathcal{M}}(X_S) \arrow[r] & H^{1,1}_{\mathcal{M}}(X_U) \arrow[r] & \mathbb{Q}^{\pi_0(X_Z)} \arrow[r] & H^{2,1}_{\mathcal{M}}(X_S) \arrow[r] & H^{2,1}_{\mathcal{M}}(X_U) \arrow[r] & 0
\end{tikzcd}$$

We then pass to the limit over all non-empty sets and use continuity for $\mathbf{DA}$. We obtain that $\mathrm{HP}^{n-2}(S) \to \mathrm{HP}^{n-2}(\kappa(S))$ is an isomorphism for $n \neq 1, 2$, and we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{HP}^{-1}(X/S) \arrow[r] \arrow[d] & \mathrm{HP}^{-1}(X_{\kappa(S)}/\kappa(S)) \arrow[r] \arrow[d] & \mathbb{Q}^{\pi_0(X_Z)} \arrow[r] \arrow[d, equal] & \mathrm{HP}^0(X/S) \arrow[r] \arrow[d] & \mathrm{HP}^0(X_{\kappa(S)}/\kappa(S)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{1,1}_{\mathcal{M}}(X_S) \arrow[r] & H^{1,1}_{\mathcal{M}}(X_{\kappa(S)}) \arrow[r] & \mathbb{Q}^{\pi_0(X_Z)} \arrow[r] & H^{2,1}_{\mathcal{M}}(X_S) \arrow[r] & H^{2,1}_{\mathcal{M}}(X_{\kappa(S)}) \arrow[r] & 0.
\end{tikzcd}$$

Applying the already established result in the field case (for the function field $\kappa(S)$) and the five lemma completes the proof.

□

In the special case of a relative curve, we can remove the regularity hypothesis on the base. This yields a general computation of the motive of a smooth projective curve over any base.

Corollary 3.3.11. *Let $f : C \to S$ be a smooth projective curve. The morphism*

$$\Theta_f : (\Sigma^\infty \mathrm{P}(C/S)_{\mathbb{Q}})(-1)[-2] \to f_*\mathbb{Q}_C$$

is an isomorphism, and induces an isomorphism

$$\Sigma^\infty \mathrm{P}(C/S) \simeq M_S(C)$$

If f has a section $s : S \to C$, we have, moreover, an isomorphism

$$f_\sharp \mathbb{Q}_C \simeq \mathbb{Q}_S \oplus \Sigma^\infty \operatorname{Jac}(C/S) \oplus \mathbb{Q}_S(1)[2].$$

Proof. Let us prove that Θ_f is an isomorphism. By Lemma 2.4.2, it is enough to show that $s^*\Theta_f$ is an isomorphism for any $s \in S$. By Proposition 3.2.39 and Proposition 3.2.36 we are then reduced to the case when S is the spectrum of a field. This is then a special case of Theorem 3.3.1.

The last statement then follows by using the section to split up the distinguished triangles coming in the structure of $\mathrm{P}(C/S)$. □

3.3.3 Finiteness and applications

As an application of the computation in the previous section, we can now prove a fundamental finiteness result for ω^1.

Theorem 3.3.12. *Let S be a noetherian finite-dimensional excellent scheme. Assume that S admits the resolution of singularities by alterations. Then the functor $\omega^1 : \mathbf{DA}^{\mathrm{coh}}(S) \to \mathbf{DA}^1(S)$ preserves compact objects.*

Proof. We follow the argument of [AZ12, Proposition 2.14 (vii)] for the case of ω^0, with minor changes.

By Corollary 3.1.19 (ii) we can assume that S is reduced. We prove the result by noetherian induction on S. Let M be in $\mathbf{DA}^{\mathrm{coh}}_c(S)$. Since M is compact and cohomological, Lemma 3.1.8 implies that there exists a finite family $\{f_i\}_{i=1}^n$ of proper morphisms $f_i : X_i \to S$ such that M lies in the triangulated subcategory generated by the motives $f_{i*}\mathbb{Q}_{X_i}$. By Proposition 3.2.27, there exists an everywhere dense open subset $U \subset S$ such that $f_i \times_S U$ is Pic-smooth for every i. We can moreover assume that U is regular. Write $j : U \to S$ for the open immersion and $i : Z \to S$ for the complementary reduced closed immersion. By Proposition 3.1.11, because of the hypothesis on S, the colocalisation triangle

$$i_* i^! M \to M \to j_* j^* M \xrightarrow{+}$$

lies in $\mathbf{DA}^{\mathrm{coh}}(S)$. We apply ω^1 and use Proposition 3.3.3 (iii) to obtain a distinguished triangle

$$i_* \omega^1(i^! M) \to M \to \omega^1(j_* j^* M) \xrightarrow{+} .$$

By induction, we know that $\omega^1(i^! M)$ is compact, so it is enough to show that $\omega^1(j_* j^* M)$ is as well. By Proposition 3.3.3 (iii), we have an isomorphism $\omega^1(j_* j^* M) \simeq \omega^1(j_* \omega^1 M)$. Put $N = j_* \omega^1(j^* M)$; we have to show that $\omega^1(N)$ is compact. The motive $j^* M$ lies in the triangulated subcategory generated by the motives $(f_i \times_S U)_* \mathbb{Q}$ with $f_i \times_S U$ Pic-smooth and U regular, hence by Theorem 3.3.1 we have $\omega^1(j^* M)$ compact. This implies that N is compact, with $j^* N \in \mathbf{DA}^1(U)$. In particular, we have $j_! j^* N \in \mathbf{DA}^1_c(S)$. Thus applying ω^1 to the localisation triangle for N and using Proposition 3.3.3 (iii) yield a distinguished triangle

$$j_! j^* N \to \omega^1 N \to i_* \omega^1 i^* N \xrightarrow{+} .$$

By Proposition 3.3.3 (vi), we have $i^* \omega^1(N) \simeq \omega^1(i^* N)$, which is compact by induction. This concludes the proof. □

3.4 Motivic t-structures

We introduce the motivic t-structures on $\mathbf{DA}_1(S)$ and $\mathbf{DA}^1(S)$ and study how Deligne 1-motives fit in its heart.

3.4.1 Generated t-structures

We fix a (noetherian, finite dimensional) base scheme S for the rest of this section. We want to define t-structures by generators and relations, using the following result of Morel. We use the notations on generated subcategories of triangulated categories from the conventions section.

Proposition 3.4.1. *[Ayo07a, Lemme 2.1.69, Proposition 2.1.70] Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{G}$ be a family of compact objects in $\mathcal{T}$. Define $\mathcal{T}_{\geq 0} = \ll \mathcal{G} \gg_+$ and $\mathcal{T}_{<0}$ as the full subcategory of all objects N with*

$$\forall n \in \mathbb{N},\ \forall G \in \mathcal{G},\ \mathrm{Hom}(G, N[-n]) = 0.$$

Then $(\mathcal{T},\ \mathcal{T}_{\geq 0},\ \mathcal{T}_{<0})$ is a t-structure on $\mathcal{T}$, that we denote by $t(\mathcal{G})$ and call the t-structure generated by $\mathcal{G}$.

Before we come to the generators, we need a small discussion on connected components in the relative setting, based on [Rom11]. For $f : X \to S$ morphism of schemes, let $\pi_0(X/S)$ be the functor $\mathsf{Sch}/S \to \mathrm{Ens}$ which associates to T/S the set of all open subsets U of X_T which are faithfully flat and of finite type over S and such that for all $\bar{t} \in T$ geometric point, $U_{\bar{t}}$ is a connected component of $X_{\bar{t}}$ (this is compatible with the notation introduced after Hypothesis 3.2.21 in the smooth projective case). By [Rom11, Theoreme 2.5.2], the functor $\pi_0(X/S)$ is representable by an étale algebraic space of finite type over S. In particular $\pi_0(X/S)$ is a constructible sheaf of sets (in the sense that it comes from a constructible sheaf of sets on the small étale site). By [SGA73, Expose IX Proposition 2.5] and localisation, this shows that the motive $M_S(\pi_0(X/S)) \in \mathbf{DA}(S)$ (i.e., the motive attached to the sheaf $\mathbb{Q}(\pi_0(X/S))$) is in $\mathbf{DA}_{0,c}(S)$.

Let $X \in \mathsf{Sm}/S$. There is a natural morphism $X \to \pi_0(X/S)$ which is surjective and smooth, and an induced morphism of motives $M_S(X) \to M_S(\pi_0(X/S))$. For any smooth S-scheme X, we choose a distinguished triangle

$$M_S^{\perp}(X) \to M_S(X) \to M_S(\pi_0(X/S)) \xrightarrow{+} .$$

The construction of this triangle commutes (up to a non-canonical isomorphism) with arbitrary base change, and we will use this fact without comment below.

We can now introduce our candidate generating families. The definition uses Deligne 1-motives over a base: for definitions and notations, we refer to the first section of Appendix 3.A.

Definition 3.4.2. We define classes of objects in $\mathbf{DA}(S)$ as follows. We put

$$\mathcal{CG}_S = \left\{ M_S(C), M_S^{\perp}(C)[-1] \mid \ C/S \text{ smooth curve} \right\},$$

$$\mathcal{JG}_S = \left\{ e_\sharp \Sigma^\infty (K \otimes \mathbb{Q}) | e : U \to S \text{ étale}, K = \begin{array}{c} \mathbb{Z}[V],\ V/U \text{ finite étale} \\ R_{V/S}\mathbb{G}_{\mathrm{m}}[-1],\ V/U \text{ finite étale} \\ \mathrm{Jac}(C/U)[-1],\ C/U \text{ smooth projective curve} \end{array} \right\},$$

and

$$\mathcal{DG}_S = \{ e_\sharp \mathcal{R}_U(K) |\ e : U \to S \text{ étale },\ K \in \mathcal{M}_1(U) \}.$$

We call objects in $\mathcal{CG}_S$ (resp. $\mathcal{JG}_S$, $\mathcal{DG}_S$) *curve generators* (resp. *Jacobian generators*, *Deligne generators*).

We are mostly interested in $\mathcal{CG}_S$ and $\mathcal{DG}_S$, the family $\mathcal{JG}_S$ is introduced as a technical intermediate.

Lemma 3.4.3. *The families above have the following properties.*

(i) *Let $f : T \to S$ be a morphism of schemes. Then we have $f^*\mathcal{CG}_S \subset \mathcal{CG}_T$, $f^*\mathcal{JG}_S \subset \mathcal{JG}_T$ and $f^*\mathcal{DG}_S \subset \mathcal{DG}_T$.*

(ii) *Let $e : T \to S$ be an étale morphism. Then $e_\sharp\mathcal{CG}_T \subset \mathcal{CG}_S$, $e_\sharp\mathcal{JG}_T \subset \mathcal{JG}_S$ and $e_\sharp\mathcal{DG}_T \subset \mathcal{DG}_S$.*

Proof. Point (i) follows from the $\mathrm{Ex}^*_\sharp$ isomorphism and Corollary 3.2.2. Point (ii) follows directly from the definition. □

Lemma 3.4.4. *Let S be a noetherian finite dimensional scheme. We have $\mathcal{JG}_S \subset \mathcal{DG}_S$ and $\langle \mathcal{JG}_S \rangle_{(+)} \subset \langle \mathcal{CG}_S \rangle_{(+)}$.*

Proof. The first statement follows immediately from the definition. We turn to the second one. We only need to treat the + variant.

Let $e : U \to S$ be an étale morphism and $h : V \to U$ a finite étale morphism. The motive $\mathbb{Q}[V]$ is clearly in $\mathcal{CG}_U$. Consider the smooth curve $f : \mathbb{G}^1_{\mathrm{m}} \times V \to U$; we have $M_U^{\perp}(\mathbb{G}^1_{\mathrm{m}} \times V)[-1] \simeq \Sigma^\infty R_{V/U}\mathbb{G}_{\mathrm{m}} \otimes [-1]$, which shows that $\Sigma^\infty R_{V/U}\mathbb{G}_{\mathrm{m}} \otimes [-1]$ is in $\mathcal{CG}_U$. Let $e : U \to S$ be an étale morphism and be $f : C \to U$ a smooth projective curve. By Corollary 3.3.11, we have an isomorphism $M_U(C) \simeq \Sigma^\infty \mathrm{P}(C/U)$. The Picard complex of the curve C fits into distinguished triangles

$$R_{\pi_0(f)}\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q} \to \mathrm{P}(C/U)_{\mathbb{Q}} \to \mathcal{P}ic^{\mathrm{sm}}_{C/U} \otimes \mathbb{Q} \xrightarrow{+}$$

and

$$\mathrm{Jac}(C/U) \otimes \mathbb{Q} \to \mathcal{P}ic^{\mathrm{sm}}_{C/U} \otimes \mathbb{Q} \to \mathbb{Q}[\pi_0(C/U)] \xrightarrow{+}.$$

Moreover, the map $M_U(C) \to M_U(\pi_0(C/U))$ coincides modulo the isomorphism above with the composite map $\Sigma^\infty \mathrm{P}(C/U)_{\mathbb{Q}} \to \Sigma^\infty \mathbb{Q}[\pi_0(C/U)]$. This gives us a distinguished triangle

$$M_U^{\perp}(C)[-1] \to \Sigma^\infty \mathrm{Jac}(C/U) \otimes \mathbb{Q}[-1] \to \Sigma^\infty R_{\pi_0(f)}\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q} \xrightarrow{+}$$

which combined with the previous arguments shows that $\mathrm{Jac}(C/U) \otimes \mathbb{Q}[-1]$ is in $\langle \mathcal{CG}_U \rangle_+$. Finally, in all three cases, we apply $e_\sharp$ and use the previous lemma. This shows that $\mathcal{JG}_S \subset \langle \mathcal{CG}_S \rangle_+$, as required. □

We now come to a more difficult stability property.

Proposition 3.4.5. *Let $i : Z \to S$ be a closed immersion. Then*

$$i_* \langle \mathcal{JG}_Z \rangle_{(+)} \subset \langle \mathcal{JG}_S \rangle_{(+)}.$$

Proof. Let $r : Z_{\mathrm{red}} \to Z$ be the canonical closed immersion. Localisation implies that $\mathrm{id} \simeq r_* r^*$. Since r^* preserves $\mathcal{JG}$ by Lemma 3.4.3, we see that it is enough to show the property for $i \circ r$. We can thus assume Z reduced.

We proceed by induction on the dimension of Z. If $\dim(Z) = 0$, because Z is reduced, it is a disjoint union of closed points of S. Then i_* is canonically the direct sum of the corresponding push-forwards for each point, so we can assume that Z is a single closed point $s \in S$.

There are three different types of generators in $\mathcal{JG}_s$. Note that over a point, the morphism e involved in the definition is a finite étale field extension, and we can always "absorb" it into the generator itself (for the case of the Jacobian of a smooth projective curve, this entails noticing that $e_\sharp \operatorname{Jac}(C/U) \simeq e_* \operatorname{Jac}(C/U) \simeq R_e \operatorname{Jac}(C/U)$ is the Weil restriction of the Jacobian of C to s, which is nothing else than the Jacobian of C considered as a smooth projective curve over s). So we assume $e = \mathrm{id}$ in what follows.

We first consider the case of a generator $\Sigma^\infty \mathbb{Q}[V] \simeq a_\sharp \mathbb{Q}$ with $a : V \to s$ a finite étale morphism. By standard spreading out results [Gro66b, §8], there exists an open neighbourhood $s \in U \overset{c}{\hookrightarrow} S$ and a finite étale morphism $\tilde{a} : \widetilde{V} \to U$ extending a, in the sense that we have a commutative diagram of schemes

$$\begin{tikzcd}
\widetilde{V}^\circ \arrow[r, "\tilde{\jmath}"] \arrow[d, "\tilde{a}^\circ"'] & \widetilde{V} \arrow[d, "\tilde{a}"'] & V \arrow[l, "\tilde{\imath}"'] \arrow[d, "a"'] \\
U \setminus s \arrow[r, "\bar{\jmath}"] & U & s \arrow[l, "\bar{\imath}"']
\end{tikzcd}$$

with cartesian squares. By localisation, we have a distinguished triangle

$$\tilde{\jmath}_! \tilde{\jmath}^* \tilde{a}_\sharp \mathbb{Q} \to \tilde{a}_\sharp \mathbb{Q} \to \tilde{\imath}_* \tilde{\imath}^* \tilde{a}_\sharp \mathbb{Q} \xrightarrow{+}$$

to which we apply $c_\sharp$ and then rewrite as

$$(c\bar{\jmath})_\sharp \tilde{a}^\circ_\sharp \mathbb{Q} \to c_\sharp \tilde{a}_\sharp \mathbb{Q} \to i_* a_\sharp \mathbb{Q} \xrightarrow{+} .$$

The motives $(c\bar{\jmath})_\sharp \tilde{a}^\circ_\sharp \mathbb{Q}$ and $c_\sharp \tilde{a}_\sharp \mathbb{Q}$ are in $\mathcal{JG}_S$, so this triangle shows that $i_* a_\sharp \mathbb{Q}$ lies in $\langle \mathcal{JG}_S \rangle_+$.

The case of a generator of the form $\Sigma^\infty(R_{V/S}\mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}) \simeq a_\sharp \mathbb{Q}(1)[1]$ (cf. Corollary 3.2.11) for $a : V \to s$ a finite étale morphism follows from essentially the same proof, twisting by $\mathbb{Q}(1)[1]$.

We now do the case of a generator of the form $\Sigma^\infty \operatorname{Jac}(C/s)$ with $f : C \to s$ a smooth projective curve. For this, we use standard results from the deformation theory of curves. Namely, by [SGA03, Théorème 7.3, Corollaire 7.4], the curve C can be deformed to a smooth projective curve $\hat{C}$ on $\mathbf{Spec}(\hat{\mathcal{O}}_{S,s})$. By the Artin approximation theorem, one can in fact deform C to a smooth projective curve C^h on $\mathbf{Spec}(\mathcal{O}^h_{S,s})$ where $\mathcal{O}^h_{S,s}$ is the henselian local ring of S at s. Using spreading out results from [Gro66b, §8], we arrive at the following situation. We have a pointed étale neighbourhood $(c : U \to S, s)$ of (S, s) and a smooth projective curve $\tilde{f} : \widetilde{C} \to U$ which extends C.

We form the following diagram of schemes with cartesian squares

$$\begin{tikzcd}
\widetilde{C}^0 \arrow[r, "\tilde{\jmath}"] \arrow[d, "\tilde{f}^\circ"'] & \widetilde{C} \arrow[d, "\tilde{f}"'] & C \arrow[l, "\tilde{\imath}"'] \arrow[d] \\
U^\circ \arrow[r, "\bar{\jmath}"] \arrow[dr, "c^\circ"'] & U \arrow[d, "c"] & s \arrow[l, "\bar{\imath}"'] \arrow[d, equal] \\
 & S & s \arrow[l, "i"']
\end{tikzcd}$$

We have a localisation triangle

$$\bar{\jmath}_! \bar{\jmath}^* \Sigma^\infty \operatorname{Jac}(\widetilde{C}/U) \to \Sigma^\infty \operatorname{Jac}(\widetilde{C}/U) \to \bar{\imath}_! \bar{\imath}^* \Sigma^\infty \operatorname{Jac}(\widetilde{C}/U) \xrightarrow{+}$$

to which we apply $c_\sharp$ and rewrite using various base change isomorphisms to obtain

$$(c^\circ \tilde{f}^\circ)_\sharp \mathbb{Q}_{\widetilde{C}^\circ} \to (c\tilde{f})_\sharp \mathbb{Q}_{\widetilde{C}} \to i_* f_\sharp \mathbb{Q}_C \xrightarrow{+}$$

The first two terms of this complex are in $\mathcal{JG}_S$, and this shows $i_* f_\sharp \mathbb{Q}_C$ is in $\langle \mathcal{JG}_S \rangle_+$. This concludes the proof in the case $\dim(Z) = 0$.

We now come to the induction step. Let $M \in \mathcal{JG}_Z$. Write for the moment $M = e_\sharp \Sigma^\infty G \otimes \mathbb{Q}$ with G one of the three possible types and $e : U \to S$ étale.

Let $k : W \to Z$ be a dense open irreducible subset such that e_W is finite étale. Let $l : T \to Z$ be the complementary reduced closed immersion; let further $k' : W' \to S$ be an open immersion with $W' \cap Z = W$ and $l' : T' \to Z$ be the complementary reduced closed immersion. Write $m : W \to W'$ and $n : T \to R$ for the induced closed immersions.

We have a localisation triangle for k, l to which we apply $i_!$ and get

$$i_! k_! k^* M \to i_* M \to i_* l_* l^* M \xrightarrow{+}$$

which can be rewritten as

$$k'_! m_! k^* M \to i_* M \to (l' \circ n)_* l^* M \xrightarrow{+} .$$

By Lemma 3.4.3 (i) , we have $k^* M \in \mathcal{JG}_W$ and $l^* M \in \mathcal{JG}_Z$. We have $\dim(T) < \dim(Z)$ so that by induction the third term of this triangle is in $\langle \mathcal{JG}_S \rangle_+$. Moreover $k'_!$ preserves $\langle \mathcal{JG} \rangle_+$ by Lemma 3.4.3 (i). Together, this means that to show that $i_* M$ is in $\langle \mathcal{JG}_S \rangle_+$, we need only show that $m_! k^* M$ is in $\langle \mathcal{JG}_{W'} \rangle_+$. We are thus reduced to the case where Z is irreducible (with generic point η) and e a finite étale morphism. In that situation we can again "absorb" $e_\sharp$ into G and assume $e = \mathrm{id}$ and $V = S$.

The rest of the induction step consists of applying the same type of spreading out/deformation arguments we used in the $\dim(Z) = 0$ case to G_η. Since the three cases are similar and the case of $G = \mathrm{Jac}(C/S)$ with $f : C \to S$ smooth projective curve is the most complicated, we only detail that one.

By the same deformation argument as in the dimension 0 case, which applies to the non-closed point η as well, we can find a pointed étale neighbourhood $(e : W \to S, x \to \eta)$ of (S, η), a smooth projective curve $\tilde{f} : \widetilde{C} \to W$ which extends C_η.

Put $V = \overline{\{x\}} \subset W$ be the closure of x. By spreading-out, there exists an open neighbourhood $V^\circ \subset V$ of x and a dense open subset $Z^\circ \subset Z$ such that $\tilde{f}$ induces an isomorphism $V^\circ \simeq Z^\circ$ (since it is an isomorphism above η). By localisation and the induction hypothesis, we can assume that $Z^\circ = Z$. We now have a smooth projective curve above an open set of S which extends f, and we can then conclude by localisation as in the end of the proof of the $\dim(Z) = 0$ case. This finishes the proof. □

The deformation theory argument in the proof is the reason why we have introduced an arbitrary étale morphism in the definitions of $\mathcal{DG}$ and $\mathcal{JG}$, instead of say an open immersion.

We are now in position to exhibit generators for $\mathbf{DA}_1(S)$ and $\mathbf{DA}^1(S)$.

Proposition 3.4.6. *Let S be a noetherian finite-dimensional scheme.*

(i) $\langle \mathcal{CG}_S \rangle_{(+)} = \langle \mathcal{JG}_S \rangle_{(+)} = \langle \mathcal{DG}_S \rangle_{(+)}$.

(ii)
$$\mathbf{DA}_{1,c}(S) = \langle \mathcal{CG}_S \rangle = \langle \mathcal{JG}_S \rangle = \langle \mathcal{DG}_S \rangle$$

and
$$\mathbf{DA}_1(S) = \ll \mathcal{CG}_S \gg = \ll \mathcal{JG}_S \gg = \ll \mathcal{DG}_S \gg .$$

(iii)
$$\mathbf{DA}^1_c(S) = \langle \mathcal{CG}_S(-1) \rangle = \langle \mathcal{JG}_S(-1) \rangle = \langle \mathcal{DG}_S(-1) \rangle$$

and
$$\mathbf{DA}_1(S) = \ll \mathcal{CG}_S(-1) \gg = \ll \mathcal{JG}_S(-1) \gg = \ll \mathcal{DG}_S(-1) \gg .$$

Proof. Let us prove Point (i). Using Lemma 3.4.3 and localisation, we can assume that S is reduced. Lemma 3.4.3 already provides us with $\langle \mathcal{JG}_S \rangle_{(+)} \subset \langle \mathcal{DG}_S \rangle_{(+)}$ and $\langle \mathcal{JG}_S \rangle_{(+)} \subset \langle \mathcal{CG}_S \rangle_{(+)}$. We prove the other inclusions by noetherian induction on S. As usual it is enough to treat the + version. Let M be in either $\mathcal{G}_S$ or $\mathcal{DG}_S$. By Lemmas 3.4.5, 3.4.3 and localisation, to proceed with the induction, it is enough to show that there exists a non-empty open set $j : U \to S$ such that j^*M lies in $\langle \mathcal{JG}_U \rangle_+$.

We first look at $\mathcal{G}_S$. Let $f : C \to S$ be a smooth morphism of relative dimension ≤ 1. Let η be a generic point of S. If η were perfect, we could use the smooth projective completion of C_η. In general, we have to be more careful. After a finite inseparable extension $h : \mathbf{Spec}(\eta') \to \mathbf{Spec}(\eta)$, the smooth curve $C_{\eta'}$ has a smooth projective completion $\bar{C}_{\eta'}$, with complement an étale η'-scheme $\partial C_{\eta'}$. By the separation property of $\mathbf{DA}$, we have $M_\eta(C_\eta) \simeq h_* M_{\eta'}(C_{\eta'})$. By localisation applied to the pair $(\bar{C}'_\eta, C_{\eta'})$, we get a distinguished triangle

$$h_* M_{\eta'}(\partial C_{\eta'})(1)[1] \to h_* M_{\eta'}(C_{\eta'}) \to h_* M_{\eta'}(\bar{C}_{\eta'}) \xrightarrow{+} .$$

By Lemma 3.1.27, there exists a finite étale morphism $\partial\widetilde{C}/\eta$ (resp. a smooth projective curve $\widetilde{C}/\eta$) such that $h_* M_{\eta'}(\partial C_{\eta'}) \simeq M_\eta(\partial\widetilde{C})$ (resp. $h_* M_{\eta'}(\bar{C}_{\eta'}) \simeq M_\eta(\widetilde{C})$). Putting this together, we get a distinguished triangle

$$M_\eta(\partial\widetilde{C})(1)[1] \to M_\eta(C_\eta) \to M_\eta(\widetilde{C}) \xrightarrow{+} .$$

Moreover, by spreading out, we can find a normal open subset $\eta \in V \subset S$ such that $\partial\widetilde{C}$ (resp. $\widetilde{C}$) extend to a finite étale morphism (resp. a smooth projective morphism) over V and a distiguished triangle

$$M_V(\partial\widetilde{C})(1)[1] \to M_V(C_V) \to M_V(\widetilde{C}) \xrightarrow{+} .$$

This triangle, together with Corollary 3.2.11 applied to $R_{\partial\widetilde{C}/V}\mathbb{G}_\mathrm{m}$ and Corollary 3.3.11 applied to $\widetilde{C}$, shows that $M_V(C_V)$ is in $\ll \mathcal{JG}_V \gg_+$. An analysis of the construction of $\widetilde{C}$ above shows that $\pi_0(C_V/V) \simeq \pi_0(\widetilde{C}/V)$, so that we have a distinguished triangle

$$M_V(\partial\widetilde{C})(1) \to M_V^\perp(C_V)[-1] \to M_V^\perp(\widetilde{C})[-1] \xrightarrow{+} .$$

Using Corollary 3.3.11, we have

$$M_V^\perp(\widetilde{C})[-1] \to \Sigma^\infty \operatorname{Jac}(\widetilde{C}/V) \otimes \mathbb{Q}[-1] \to \Sigma^\infty R_{\pi_0(f)}\mathbb{G}_\mathrm{m} \otimes \mathbb{Q} \xrightarrow{+} .$$

which shows that $M_V^\perp(C_V)[-1]$ lies in $\ll \mathcal{JG}_V \gg_+$. We have achieved our goal.

We now look at $\mathcal{DG}_S$. A lattice (resp. a torus) on a reduced scheme is generically a direct factor of a permutation lattice (resp. torus) by [SGA70, Exp. X 6.2], while an abelian scheme on S is generically and up to isogeny a direct factor of a relative Jacobian by [Kat99, Theorem 11] applied at a generic point and a spreading out argument. This implies that for any $M \in \mathcal{DG}_S$, there exists a non-empty open $j : U \to S$ such that j^*M is a direct factor of a motive in $\mathcal{JG}_U$. This completes the proof of Point (i).

For Point (ii), we only have to notice that by definition (resp. by Lemma 3.1.8) we have $\mathbf{DA}_1(S) = \ll \mathcal{CG} \gg$ (resp. $\mathbf{DA}_{1,c}(S) = \langle \mathcal{CG}_S \rangle$) and the rest then follows from Point (i). Finally, Point (iii) is deduced from (ii) using Proposition 3.1.28. □

We come to the main definition of this chapter.

Definition 3.4.7. The *motivic t-structure* $t_{\mathbf{MM},1}(S)$ on $\mathbf{DA}_1(S)$ (resp. $t^1_{\mathbf{MM}}(S)$ on $\mathbf{DA}^1(S)$) is the t-structure $t(\mathcal{CG}_S)$ (resp. $t(\mathcal{CG}_S(-1))$). The heart of $t_{\mathbf{MM},1}$ (resp. $t^1_{\mathbf{MM}}$) is the abelian category of 1-motivic sheaves $\mathbf{MM}_1(S)$ (resp. $\mathbf{MM}^1(S)$).

The two abelian categories $\mathbf{MM}_1(S)$ and $\mathbf{MM}^1(S)$ are isomorphic via Tate twists, but embedded differently in $\mathbf{DA}(S)$. By Proposition 3.4.6, we have $t_{\mathbf{MM},1} = t(\mathcal{JG}_S) = t(\mathcal{DG}_S)$ (resp. $t^1_{\mathbf{MM}} = t(\mathcal{JG}_S(-1)) = t(\mathcal{DG}_S)$).

We now discuss some elementary exactness properties of Grothendieck operations with respect to the motivic t-structure.

Proposition 3.4.8. *The following properties hold both for $t_{\mathbf{MM},1}$ and $t^1_{\mathbf{MM}}$.*

(i) *Let f be a morphism of schemes; then f^* is t-positive.*

(ii) *Let f be a quasi-finite separated morphism; then $f_!$ is t-positive.*

(iii) *Let e be an étale morphism; then e^* is t-exact.*

(iv) *Let f be a finite morphism; then f_* is t-exact.*

The following properties hold for $t^1_{\mathbf{MM}}$.

(i) *Let f be a morphism of schemes; then $\omega^1 f_*$ is t-negative.*

(ii) *Let f be a quasi-finite separated morphism of schemes; then $\omega^1 f^!$ is t-negative.*

Proof. By Proposition 3.1.18 (resp. 3.1.17) and the very definition of ω^1, all the operations above preserve $\mathbf{DA}_1(S)$ (resp. $\mathbf{DA}^1(S)$). We prove the proposition for $t_{\mathbf{MM},1}$; the proof for $t^1_{\mathbf{MM}}$ is then obtained by twisting by $\mathbb{Q}(-1)$.

Let $f : S \to T$ be any morphism of schemes. Then f^*, $f_!$ both commute with small sums since they are left adjoints. By [Ayo07a, Lemme 2.1.78], to prove statements (i), (ii), it remains to show that $f^*\mathcal{DG}_T \subset \mathbf{DA}_1(S)_{\geq 0}$ and that when f is quasi-finite, $f_!\mathcal{DG}_S \subset \mathbf{DA}_1(S) \geq 0$.

In the case of f^*, we deduce from the $\mathrm{Ex}^*_!$ isomorphism and Proposition 3.2.4 that we have the stronger result $f^*\mathcal{DG}_T \subset \mathcal{DG}_S$. This proves (i).

For the case of $f_!$, we proceed in several steps. If e is an étale morphism, we have $e_!\mathcal{DG}_S \subset \mathcal{DG}_T$ by definition. If i is a closed immersion, we have $i_!\mathcal{DG}_S \subset \mathcal{DG}_T$ by Lemma 3.4.5. Let f be an arbitrary quasi-finite morphism. At this point, we have that for a open immersion j (resp. a closed immersion i), the functors $j_!$ and j^* (resp. the functors $i_!$ and i^*) are t-positive. This shows that to prove that an object M is t-positive, one can proceed by localisation. A noetherian induction together with the étale case above then reduce us to the case where f is finite surjective radicial, and allows us further to restricts to an arbitrary dense open set of the base. Using continuity, this reduces us to the field case, where we can apply Lemma 3.1.27. Let f be an étale morphism (resp. a finite morphism). We have seen above that f^* (resp. $f* \simeq f_!$) is t-positive. Moreover, since $e_! \simeq e_\sharp$ (resp. f^*) is t-positive, its right adjoint e^* (resp. f_*) is t-negative. This proves (iii) (resp. (iv)).

Let $f : S \to T$ be a morphism (resp. a quasi-finite separated morphism). We have seen above that $f^* : \mathbf{DA}^1(T) \to \mathbf{DA}^1(S)$ (resp. $f_! : \mathbf{DA}^1(S) \to \mathbf{DA}^1(T)$) is t-positive, so its right adjoint $\omega^1 f_*$ (resp. $\omega^1 f_!$) is t-negative. This proves (i) (resp. (ii)). □

Remark 3.4.9. To conclude this section, let us discuss the motivation behind the equivalent definitions of the motivic t-structure above. The use of Deligne 1-motives to study the motivic t-structure goes back to [Org04]. What about the curve generators? Here the story is a bit more intricate.

The paper [Ayo11] introduces among other things an approach to the motivic t-structure on $\mathbf{DM}^{\mathrm{eff}}_1(k)$ which is quite different from the approach of [Org04], [BVK10]. The idea is to start from the *homotopy* t-structure on $\mathbf{DM}^{\mathrm{eff}}(k)$, which comes from the restriction to $\mathbf{DM}^{\mathrm{eff}}(k)$ of the standard t-structure on $D(\mathbf{Sh}(\mathrm{Cor}/k, \mathbb{Q}))$ (the fact that this restriction makes sense follows from the deep results of Voevodsky on homotopy invariant presheaves with transfers). The homotopy t-structure restricts to a t-structure on $\mathbf{DM}^{\mathrm{eff}}_1(k)$; this is not completely trivial and the proof requires the functor LAlb of [ABV09]. Moreover, the homotopy t-structures on $\mathbf{DM}^{\mathrm{eff}}(k)$ and $\mathbf{DM}^{\mathrm{eff}}_1(k)$ are generated t-structures, generated respectively by $\{M^{\mathrm{eff}}_{\mathrm{tr}}(X) |\ X \in \mathsf{Sm}/k\}$ and $\{M^{\mathrm{eff}}_{\mathrm{tr}}(C) |\ C \in \mathsf{Sm}/k,\ \dim_k(C) \leq 1\}$. The motivic t-structure on $\mathbf{DM}^{\mathrm{eff}}_1(k)$ is then obtained by "perverting" the homotopy t-structure along 0-motives (see [Ayo11, Definition 3.5]; in fact Ayoub considers a perversion of the homotopy t-structure on the entire $\mathbf{DM}^{\mathrm{eff}}(k)$, which produces a t-structure on $\mathbf{DM}^{\mathrm{eff}}(k)$, and we are claiming that this perverted t-structure restricts to $\mathbf{DM}^{\mathrm{eff}}_1(k)$ (a fact which is again proved with LAlb). By construction, it comes with a nice family of generators, namely $\{M^{\mathrm{eff}}_{\mathrm{tr}}(C), M^{\mathrm{eff},\perp}_{\mathrm{tr}}(C)[-1] |\ C \in \mathsf{Sm}/k,\ \dim_k(C) \leq 1\}$.

This approach to 1-motives is concise and categorical, and it is tempting to try to generalize it to higher dimensions (replacing $\mathbf{DM}^{\mathrm{eff}}$ by $\mathbf{DA}$ to make use of the six operation formalism). There are however immediate difficulties: the homotopy t-structure of Voevodsky does not exist on an higher dimensional base [Ayo06] and the inclusion $\mathbf{DA}_1(S) \to \mathbf{DA}_{\mathrm{hom}}(S)$ does not seem to have a left adjoint which would play the role of LAlb. However the generating family still makes sense, and inspired the definition of $\mathcal{CG}$.

3.4.2 Morphisms and the Heart

In this section, we compute a number of morphism groups between objects in $\mathbf{DA}_1(S)$ and $\mathbf{DA}^1(S)$ and deduce properties of the motivic t-structure. We first recast a result of Orgogozo [Org04] in our context.

Proposition 3.4.10. *Let k be a field, $M_1, M_2 \in \mathcal{M}_1(k)$ and $n \in \mathbb{Z}$. Then*

$$\begin{aligned} \mathbf{DA}(k)(\mathcal{R}M_1, \mathcal{R}M_2[n]) &\simeq \operatorname{Ext}^n_{\mathcal{M}_1(k)}(M_1, M_2) \\ &\simeq 0, \ n \neq 0, 1. \end{aligned}$$

Proof. Let k be a perfect field. For $M \in \mathcal{M}_1(k)$, write M^{tr} for the complex of sheaves with transferts and rational coefficients attached to M in [Org04, §3.3.2]. By [Org04, Proposition 3.3.3], we have

$$\mathbf{DM}^{\mathrm{eff}}(k)(M_1^{\mathrm{tr}}, M_2^{\mathrm{tr}}[n]) \simeq \operatorname{Ext}^n_{\mathcal{M}_1(k)}(M_1, M_2)$$

which is 0 if $n \neq 0, 1$ by [Org04, Proposition 3.2.4]. So the problem consists in transferring these results to $\mathbf{DA}(k)$ and removing the restriction k perfect.

By the cancellation theorem [Voe10], the same result holds in $\mathbf{DM}(k)$ for $\Sigma^\infty_{\mathrm{tr}} M_1^{\mathrm{tr}}$ and $\Sigma^\infty_{\mathrm{tr}} M_2^{\mathrm{tr}}$. By [AHPL14, Proposition 2.10], we have $\Sigma^\infty_{\mathrm{tr}} M_i^{\mathrm{tr}} \simeq a_{\mathrm{tr}} \Sigma^\infty M_i$ for $i \in \{1, 2\}$. By [CDb, Corollary 16.2.22], a_{tr} is an equivalence of categories, hence we deduce the result in $\mathbf{DA}(k)$ for a perfect field k. Finally, using separation and Proposition 3.2.4 implies the result for a general field k. □

Corollary 3.4.11. *Let k be a field. The t-structures $t_{\mathbf{MM},1}$ (resp. $t^1_{\mathbf{MM}}$) restrict to the subcategory $\mathbf{DA}_{1,c}(k)$ (resp. $\mathbf{DA}^1_c(k)$).*

Moreover, the functor

$$\Lambda_1 : D^b(\mathcal{M}_1(k^{\mathrm{perf}})) \to \mathbf{DA}_{1,c}(k), \ K \mapsto (\mathbf{Spec}(k^{\mathrm{perf}}) \to \mathbf{Spec}(k))_* \Sigma^\infty \operatorname{Tot}(K)$$

(resp.

$$\Lambda^1 : D^b(\mathcal{M}_1(k^{\mathrm{perf}})) \to \mathbf{DA}^1_c(k), \ K \mapsto \Sigma^\infty \operatorname{Tot}(K)(-1))$$

is an equivalence of triangulated categories with t-structures.

Proof. The statement for $t^1_{\mathbf{MM}}$ is deduced from the one for $t_{\mathbf{MM},1}$ by twisting, so we only prove the $t_{\mathbf{MM},1}$ version. The functor Λ_1 in the statement is fully faithful by separation and Proposition 3.4.10. Moreover, the same proposition implies that for all $M_1, M_2 \in \mathcal{M}_1(k)$ and $n < 0$, we have $\mathbf{DA}(k)(\Sigma^\infty M_1, \Sigma^\infty M_2[n]) = 0$. We deduce that $\Sigma^\infty M_2$ is t-negative; since it is t-positive by definition, it is in $\mathbf{MM}_{1,c}(k)$. From this one deduces that the functor Λ_1 is t-exact. To conclude the proof, it remains to show that Λ_1 is essentially surjective. This follows from the fact that $\mathbf{DA}_{1,c}(k)$ is compactly generated by motives of smooth projective curves (Proposition 3.1.26) and the computation of the motive of such a curve (Proposition 3.2.8). □

The following theorem shows the advantage of the Deligne generating family: it lies in the heart of the motivic t-structure.

Theorem 3.4.1. *We have $\mathcal{DG}_S \subset \mathbf{MM}_1(S)$ (resp. $\mathcal{DG}_S(-1) \subset \mathbf{MM}^1(S)$).*

Proof. We have shown in Proposition 3.4.6 that the generators are t-positive, it remains to show that they are t-negative. This translates into the following vanishing statement. Let S be a noetherian finite dimensional scheme. Let $\pi : C \to S$ be a smooth curve. Let $M = f_! \mathcal{R}\mathbb{M} \in \mathcal{DG}_S$ (i.e., $f : V \to S$ étale, $\mathbb{M} \in \mathcal{M}_1^{\mathrm{pure}}(V)$). Then

$$\forall n < 0, \ \mathbf{DA}(S)(M_S(C), M[n]) = 0 \tag{$\mathcal{V}$}$$

and

$$\forall n < 0, \ \mathbf{DA}(S)(M_S^\perp(C)[-1], M[n]) = 0 \tag{$\mathcal{V}^\perp$}$$

For this, we will study the long exact sequence

$$\ldots \to \mathbf{DA}(S)(M_S(\pi_0(C/S)), M[n]) \to \mathbf{DA}(S)(M_S(C), M[n]) \to \mathbf{DA}(S)(M_S^\perp(C), M[n]) \to \ldots \tag{E}$$

for various Deligne 1-motives $\mathbb{M}$.

By Zariski's main theorem, there exists a factorisation $f = \bar{f} \circ j$ with $\bar{f} : \bar{V} \to S$ finite and $j : V \to \bar{V}$ everywhere dense open immersion. Combining this with the $(\bar{f}^*, \bar{f}_*)$ adjunction, the $\mathrm{Ex}_!^*$ isomorphism and Proposition 3.2.4, we get a long exact sequence

$$\begin{tikzcd} \ldots \arrow[r] & \mathbf{DA}(\bar{V})(M_{\bar{V}}(\pi_0(C_{\bar{V}}/\bar{V})), j_!\mathcal{R}\mathbb{M}[n]) \arrow[r] & \mathbf{DA}(\bar{V})(M_{\bar{V}}(C_{\bar{V}}), j_!\mathcal{R}\mathbb{M}[n]) \arrow[d] \\ & \ldots & \mathbf{DA}(\bar{V})(M_{\bar{V}}^{\perp}(\bar{V}), j_!\mathcal{R}\mathbb{M}[n]) \arrow[l] \end{tikzcd}$$

This shows we can assume $f = j$, an everywhere dense open immersion. We write $i : Z \to S$ for the complementary reduced closed immersion. By localisation and Proposition 3.2.4 again, we can assume that S is reduced.

For the rest of the proof, we look separately at the three types of pure Deligne 1-motives. We want to prove $(\mathcal{V})$ and $(\mathcal{V}^{\perp})$ by induction on the dimension of S. In each case, to treat the case of $\dim(S) = 0$, we reduced immediately to the case of $\mathbf{Spec}(k)$ for k a field, we use the $\mathcal{DG}_k$ family of generators instead of $\mathcal{G}_k$, and we apply Proposition 3.4.10. We are thus left with the induction step.

Let $\mathbb{M}$ be $[L \to 0]$ with L a lattice on V. We prove $(\mathcal{V})$ and $(\mathcal{V}^{\perp})$ by induction on the dimension of S.

Let $l : W \to S$ an everywhere dense open immersion with $W \to V$ and $k : Y \to S$ the complementary reduced closed immersion. Then by localisation we have short exact sequences

$$\mathbf{DA}(S)(M_S(C), l_!l^*M[n]) \to \mathbf{DA}(S)(M_S(C), M[n]) \to \mathbf{DA}(M_S(C), k_*k^*M[n])$$

and

$$\mathbf{DA}(S)(M_S^{\perp}(C)[-1], l_!l^*M[n]) \to \mathbf{DA}(S)(M_S^{\perp}(C)[-1], M[n]) \to \mathbf{DA}(M_S^{\perp}(C)[-1], k_*k^*M[n])$$

and in both cases the right term vanishes for $n < 0$ by the (k^*, k_*)-adjunction and the induction hypothesis (since $\dim(Z) < \dim(S)$). This means we can replace M with

$$l_!l^*M \simeq l_!l^*j_!\mathcal{R}\mathbb{M} \simeq (W \to S)_!(W \to V)^*\mathcal{R}\mathbb{M} \simeq (W \to S)_!\mathcal{R}\mathbb{M}_W$$

where we have used the $\mathrm{Ex}_!^*$ isomorphism and Corollary 3.2.2. In other words, we can replace the dense open V by any smaller dense open W.

Using this reduction, we can assume V to be normal. This allows us to write $\mathcal{R}\mathbb{M}$ as a direct factor of $e_*\mathbb{Q}$ for a finite étale morphism $e : T \to V$. Applying Zariski's main theorem to the morphism $j \circ e : T \to S$ and adjunction, we reduce to the case $M = \mathbb{Q}_V$. The advantage of this reduction is that M then extends to a motive on S, namely $\mathbb{Q}_S$. By localisation, we have exact sequences

$$\mathbf{DA}(S)(M_S(C), i_*\mathbb{Q}[n-1]) \to \mathbf{DA}(S)(M_S(C), j_!\mathbb{Q}[n]) \to \mathbf{DA}(S)(M_S(C), \mathbb{Q}[n])$$

and

$$\mathbf{DA}(S)(M_S^{\perp}(C)[-1], i_*\mathbb{Q}[n-1]) \to \mathbf{DA}(S)(M_S^{\perp}(C)[-1], j_!\mathbb{Q}[n]) \to \mathbf{DA}(M_S^{\perp}(C)[-1], \mathbb{Q}[n]),$$

and in both cases the left term vanishes for $n < 0$ by adjunction and induction. This means we can assume $V = S$.

After all these reductions, the long exact sequence (E) can be written as

$$\ldots \to \mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}[n]) \to \mathbf{DA}(S)(M_S(C), \mathbb{Q}[n]) \to \mathbf{DA}(S)(M_S^{\perp}(C), \mathbb{Q}[n]) \to \ldots$$

By adjunction and Proposition 3.B.3 (i), we get $\mathbf{DA}(S)(M_S(C), \mathbb{Q}[n]) = 0$ for $n < 0$. By Proposition 3.B.3 (iv) applied to $C \to \pi_0(C/S)$, which is smooth with geometrically connected fibers, we have $\mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}) \simeq \mathbf{DA}(S)(M_S(C), \mathbb{Q})$ and $\mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}[1] \simeq \mathbf{DA}(S)(M_S(C), \mathbb{Q}[1])$ (to be precise, one has to apply étale descent because $\pi_0(C/S)$ is only an algebraic space). Together, this shows $(\mathcal{V})$ and $(\mathcal{V}^{\perp})$.

Let now $\mathbb{M}$ be of the form $[0 \to T]$ with T a torus on V. The proof is quite similar to the lattice case.

As in the proof for a lattice, we can replace the dense open V by any smaller dense open. Again, this lets us assume that V is normal, hence reduce to a permutation torus, then finally to $T = \mathbb{G}_{\mathrm{m}}$. Then $\mathcal{R}\mathbb{M} \simeq \mathbb{Q}_V(1)$ extends to a motive on S, namely $\mathbb{Q}_S(1)$. By localisation, we have distinguished triangles

$$\mathbf{DA}(S)(M_S(C), i_*\mathbb{Q}(1)[n-1]) \to \mathbf{DA}(S)(M_S(C), j_!\mathbb{Q}(1)[n]) \to \mathbf{DA}(S)(M_S(C), \mathbb{Q}(1)[n])$$

and

$$\mathbf{DA}(S)(M_S^{\perp}(C)[-1], i_*\mathbb{Q}(1)[n]) \to \mathbf{DA}(S)(M_S^{\perp}(C)[-1], j_!\mathbb{Q}(1)[n]) \to \mathbf{DA}(M_S^{\perp}(C)[-1], \mathbb{Q}(1)[n]),$$

and in both cases the left term vanishes for $n < 0$ by adjunction and induction. This means we can assume $V = S$.

After these reductions, we have a long exact sequence

$$\ldots \to \mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}(1)[n]) \to \mathbf{DA}(S)(M_S(C), \mathbb{Q}(1)[n]) \to \mathbf{DA}(S)(M_S^{\perp}(C), \mathbb{Q}(1)[n]) \to \ldots$$

By adjunction and Proposition 3.B.4 (i), we have $\mathbf{DA}(S)(M_S(C), \mathbb{Q}(1)[n]) = 0$ and $\mathbf{DA}(S)(M_S(\pi_0(C/S)), \mathbb{Q}(1)[n]) = 0$ for all $n \leq 0$. This implies $(\mathcal{V})$ and $(\mathcal{V}^{\perp})$ in this case.

Let $\mathbb{M}$ finally be of the form $[0 \to A]$ with A an abelian scheme on V.

As in the two previous cases, we can replace the dense open V by any smaller dense open. Using [Kat99, Theorem 11] and continuity, this lets us assume that there exists a smooth projective curve $f : D \to V$ together with a section $s : V \to D$ such that the $\mathcal{R}[0 \to A]$ is a direct factor of $\mathcal{R}[0 \to \mathrm{Jac}(D/V)]$. In the following, we replace A by $\mathrm{Jac}(D/V)$.

Unlike in the two previous cases, we cannot ensure that the curve D extends to a smooth projective curve over S, so we have to work a little around this. From Corollary 3.3.11, we have an isomorphism $f_\sharp\mathbb{Q}_D \simeq \mathbb{Q}_V \oplus \Sigma^\infty \mathrm{Jac}(D/V)_{\mathbb{Q}} \oplus \mathbb{Q}_V(1)[2]$; hence $\mathcal{R}\mathbb{M} \simeq \Sigma^\infty \mathrm{Jac}(D/V)_{\mathbb{Q}}[-1]$ is a direct factor of $f_\sharp\mathbb{Q}_D[-1]$. By relative purity, we have $f_\sharp\mathbb{Q}_D[-1] \simeq f_!\mathbb{Q}_D(1)[1]$.

We apply Nagata's theorem [Nag63] [Con07] to compactify f over S: there exists an open immersion $\bar{\jmath} : D \to \overline{D}$ and a proper morphism $\bar{f} : \overline{D} \to S$ with $j \circ f = \bar{f} \circ \bar{\jmath}$. Write $\bar{\imath} : Y \to \bar{D}$ for the complementary closed immersion; note that because f was proper over V, we can choose the compactification $\bar{D}$ so that Y lies entirely over Z. This implies that $j_! f_! \simeq \bar{f}_! \bar{\jmath}_! \simeq \bar{f}_* \bar{\jmath}_!$; hence $j_! f_! \mathbb{Q}_D(1)[1] \simeq \bar{f}_* \bar{\jmath}_! \mathbb{Q}_D(1)[1]$. The motive $\bar{\jmath}_!\mathbb{Q}_D(1)[1]$ extends to a motive on $\bar{D}$, namely $\mathbb{Q}_{\bar{D}}(1)[1]$. By localisation, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{DA}(\overline{D})(M_{\overline{D}}(\pi_0(C_{\overline{D}}/\overline{D})), \bar{\imath}_*\mathbb{Q}(1)[n]) \arrow[r] \arrow[d] & \mathbf{DA}(\overline{D})(M_{\overline{D}}(C_{\overline{D}}), \bar{\imath}_*\mathbb{Q}(1)[n]) \arrow[d] \\
\mathbf{DA}(\overline{D})(M_{\overline{D}}(\pi_0(C_{\overline{D}}/\overline{D})), \bar{\jmath}_!\mathbb{Q}(1)[n+1]) \arrow[r] \arrow[d] & \mathbf{DA}(\overline{D})(M_{\overline{D}}(C_{\overline{D}}), \bar{\jmath}_!\mathbb{Q}(1)[n+1]) \arrow[d] \\
\mathbf{DA}(\overline{D})(M_{\overline{D}}(\pi_0(C_{\overline{D}}/\overline{D})), \mathbb{Q}(1)[n+1]) \arrow[r] & \mathbf{DA}(\overline{D})(M_{\overline{D}}(C_{\overline{D}}), \mathbb{Q}(1)[n+1]).
\end{tikzcd} \tag{L}$$

and for $n < 0$, the groups on the top and bottom line vanish by the $((-)_\sharp, (-)^*)$ adjunction and Proposition 3.B.4 (i) (applying étale descent for $\mathbf{DA}$ to get around the fact that $\pi_0(C/S)$ is only an algebraic space). Using that $j_!\mathcal{R}\mathbb{M}$ is a direct factor of $\bar{f}_*\bar{\jmath}_!\mathbb{Q}_D(1)[1]$, this establishes $(\mathcal{V})$ for all $n < 0$. Plugging this back in the sequence (E), we also get $(\mathcal{V}^{\perp})$ for $n < -1$. However for $n = -1$, we cannot conclude directly; rather, combining the established vanishing results and (E), we get a sequence

$$0 \to \mathbf{DA}(S)(M_S^{\perp}(C)[-1], j_!\mathcal{R}\mathbb{M}[-1]) \to \mathbf{DA}(S)(M_S(\pi_0(C/S)), j_!\mathcal{R}\mathbb{M}) \to \mathbf{DA}(S)(M_S(C), j_!\mathcal{R}\mathbb{M})$$

and we have to show that the last morphism is injective. Because of the direct factor argument, it suffices to show that the same morphism for $\bar{f}_*\bar{\jmath}_!\mathbb{Q}_D(1)[1]$ is injective. Specializing diagram (L)

for $n = 0$, we get

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \mathbf{DA}(\overline{D})(M_{\overline{D}}(\pi_0(C_{\overline{D}}/\overline{D})), \bar{\jmath}_!\mathbb{Q}(1)[1]) \arrow[r] \arrow[d] & \mathbf{DA}(\overline{D})(M_{\overline{D}}(C_{\overline{D}}), \bar{\jmath}_!\mathbb{Q}(1)[1]) \arrow[d] \\ \mathbf{DA}(\overline{D})(M_{\overline{D}}(\pi_0(C_{\overline{D}}/\overline{D})), \mathbb{Q}(1)[1]) \arrow[r] & \mathbf{DA}(\overline{D})(M_{\overline{D}}(C_{\overline{D}}), \mathbb{Q}(1)[1]). \end{tikzcd}$$

which shows that we only need to show the injectivity of

$$\mathbf{DA}(\overline{D})(M_{\overline{D}}(\pi_0(C_{\overline{D}}/\overline{D})), \mathbb{Q}(1)[1]) \to \mathbf{DA}(\overline{D})(M_{\overline{D}}(C_{\overline{D}}), \mathbb{Q}(1)[1]).$$

Applying cohomological h-descent for $\mathbf{DA}$ to a proper regular hypercovering of $\overline{D}$ (constructed using alterations after a further reduction to $\overline{D}$ of finite type over a Dedekind ring as in the proof of Proposition 3.B.3 (iv)), we see that we can assume $\overline{D}$ to be regular. Then the morphism above can be identified using Proposition 3.B.4 (ii) with the natural morphism

$$\mathcal{O}^\times(\pi_0(C_{\overline{D}}/\overline{D})) \otimes \mathbb{Q} \to \mathcal{O}^\times(C_{\overline{D}}) \otimes \mathbb{Q}.$$

Since the morphism $C_{\overline{D}} \to \pi_0(C_{\overline{D}}/\overline{D})$ is surjective and $\pi_0(C_{\overline{D}}/\overline{D})$ is reduced, the induced morphism on global functions is injective. This completes the proof of the injectivity, hence the proof of $(\mathcal{V}^\perp)$ in the missing case $n = -1$.

□

We now compute more precisely the morphisms between Deligne 1-motives over a regular base.

Theorem 3.4.2. *Let S be a regular scheme, $\mathbb{M}_1, \mathbb{M}_2 \in \mathcal{M}_1(S)$, $n \in \mathbb{Z}$. Then*

$$\mathbf{DA}(S)(\mathcal{R}\mathbb{M}_1, \mathcal{R}\mathbb{M}_2[n]) \simeq \begin{cases} 0, \ n < 0 \\ \mathcal{M}_1(S)(\mathbb{M}_1, \mathbb{M}_2), \ n = 0 \\ 0, \ n \geq 3. \end{cases}$$

In particular, the functor $\mathcal{R} : \mathcal{M}_1(S) \to \mathbf{MM}_1(S)$ is fully faithful.

Proof. By considering the connected components, we reduce immediately to the case where S is irreductible. The idea of the proof is that in the range we are considering everything happens at the generic point η. Let $j : U \to S$ closed immersion with $U \neq \emptyset$. The restriction functor $j^* : \mathcal{M}_1(S) \to \mathcal{M}_1(U)$ is fully faithful by Proposition 3.A.10. Moreover the category $\mathcal{M}_1(\eta)$ is the 2-colimit of the $\mathcal{M}_1(U)$ for U running through all non-empty open sets of S by Proposition 3.A.9. This implies that $\mathcal{M}_1(S)(\mathbb{M}_1, \mathbb{M}_2) \simeq \mathcal{M}_1(\eta)(\eta^*\mathbb{M}_1, \eta^*\mathbb{M}_2)$. On the $\mathbf{DA}(-)$ side, by continuity and Proposition 3.2.4, we have that $\mathbf{DA}(\eta)(\eta^*\mathcal{R}\mathbb{M}_1, \eta^*\mathcal{R}\mathbb{M}_2[n]) \simeq \mathrm{Colim}_{U\neq\emptyset}\, \mathbf{DA}(U)(j^*\mathcal{R}\mathbb{M}_1, j^*\mathcal{R}\mathbb{M}_2[n])$. Furthermore, by Proposition 3.4.10, we have an isomorphism

$$\mathbf{DA}(\eta)(\mathcal{R}\eta^*\mathbb{M}_1, \mathcal{R}\eta^*\mathbb{M}_2[n]) \simeq \mathrm{Ext}^n_{\mathcal{M}_1(\eta)}(\mathbb{M}_1, \mathbb{M}_2) \simeq 0$$

for $n \neq 0, 1$.

Putting everything together, we see that the statement of the proposition follows from the claim that $j^* : \mathbf{DA}(S)(\mathcal{R}\mathbb{M}_1, \mathcal{R}\mathbb{M}_2[n]) \to \mathbf{DA}(U)(j^*\mathcal{R}\mathbb{M}_1, j^*\mathcal{R}\mathbb{M}_2[n])$ is bijective for $n \neq 1, 2$. Write $i : Z \to S$ for the reduced complementary closed immersion of U in S. Consider the localisation exact sequence

$$\begin{tikzcd} \ldots \arrow[r] & \mathbf{DA}(Z)(i^*\mathcal{R}\mathbb{M}_1, i^!\mathcal{R}\mathbb{M}_2[n]) \arrow[r] & \mathbf{DA}(S)(\mathcal{R}\mathbb{M}_1, \mathcal{R}\mathbb{M}_2[n]) \arrow[d, "j^*"] \\ \ldots & \mathbf{DA}(Z)(i^*\mathcal{R}\mathbb{M}_1, i^!\mathcal{R}\mathbb{M}_2[n+1]) \arrow[l] & \mathbf{DA}(U)(j^*\mathcal{R}\mathbb{M}_1, j^*\mathcal{R}\mathbb{M}_2[n]) \arrow[l] \end{tikzcd}$$

We have to prove the vanishing of $\mathbf{DA}(Z)(i^*\mathcal{R}\mathbb{M}_1, i^!\mathcal{R}\mathbb{M}_2[n+1])$ for $n \neq 2$. By Proposition 3.2.2, we have $i^*\mathcal{R}\mathbb{M}_1 \simeq \mathcal{R}\mathbb{M}_{1,Z}$. Stratifying Z by regular constructible subschemes and applying further localisations, we can reduce to the case where Z is also regular of some codimension $1+e$ with $e \geq 0$. By absolute purity, we then have $i^!\mathcal{R}\mathbb{M}_2[n+1] \simeq i^*\mathcal{R}\mathbb{M}_2(-1-e)[n-1-2e] \simeq \mathcal{R}\mathbb{M}_{2,Z}(-1-e)[n-1-2e]$. We know from Corollary 3.2.16 that the motive $\mathcal{R}\mathbb{M}_{1,Z}(-1)$ lies in $\mathbf{DA}^1(S)$, hence we have an isomorphism

$$\mathbf{DA}(Z)(\mathcal{R}\mathbb{M}_{1,Z}, \mathcal{R}\mathbb{M}_{2,Z}(-1-e)[n-1-2e]) \simeq \mathbf{DA}(Z)(\mathcal{R}\mathbb{M}_{1,Z}(-1), \omega^1(\mathcal{R}\mathbb{M}_{2,Z}(-1)(-1-e)[n-1-2e]))$$

The motive $\mathcal{R}\mathbb{M}_{2,Z}(-1)$ is cohomological, so by Corollary 3.3.7 the group on the right hand side vanishes unless $e = 0$. If $e = 0$, we have further $\omega^1(\mathcal{R}\mathbb{M}_{2,Z}(-1)(-1)) \simeq \omega^0(\mathcal{R}\mathbb{M}_{2,Z}(-1))(-1)$. This last group was computed in Proposition 3.3.8 (iii) and we get

$$\omega^0(\mathcal{R}\mathbb{M}_{2,Z}(-1))(-1) \simeq \Sigma^\infty X_*(W_{-2}\mathbb{M}_{2,Z})_\mathbb{Q}(-1).$$

To sum up, we have reduced to show that for S regular, $\mathbb{M} \in \mathcal{M}_1(S)$ and L lattice over S, the morphism group $\mathbf{DA}(S)(\mathcal{R}\mathbb{M}, \mathcal{R}L_\mathbb{Q}[n-1])$ is 0 for $n \neq 2$. Since S is normal, we can write L as a direct factor up to isogeny of a permutation lattice. This implies that the motive $\Sigma^\infty L_\mathbb{Q}$ is a direct factor of $e_*\mathbb{Q}$ for $e : T \to S$ finite étale. By adjunction, we are then reduced to the case $L = \mathbb{Z}$. Write $\mathbb{M} = [N \to G]$ with N a lattice and G an abelian-by-torus scheme. We have a distinguished triangle

$$\mathcal{R}[0 \to G] \to \mathcal{R}\mathbb{M} \to \mathcal{R}[N \to 0] \xrightarrow{+}$$

which shows that we can treat separately the cases $\mathbb{M} = [N \to 0]$ and $\mathbb{M} = [0 \to G]$.

In the case $\mathbb{M} = [N \to 0]$, we again write N as a direct factor of a permutation lattice, which implies that $\mathcal{R}\mathbb{M}$ is a direct factor of $e'_\sharp\mathbb{Q}$ with $e' : T' \to S$ finite étale. By the $(h_\sharp, h^*)$ adjunction, we are then reduced to a computation of weight zero motivic cohomology on a regular scheme, which vanishes exactly for $n \neq 2$ by Propositions 3.B.2 and 3.B.3.

In the second case, we have $\mathcal{R}\mathbb{M} = \Sigma^\infty G_\mathbb{Q}[-1]$, which by Theorem 2.3.3 is a direct factor of $M_S(G)$. We are then done using the $((G \to S)_\sharp, (G \to S)^*)$ adjunction and Propositions 3.B.2 and 3.B.3. ☐

Remark 3.4.12. This leaves open the determination of the morphisms groups $\mathbf{DA}(S)(\mathcal{R}\mathbb{M}_1, \mathcal{R}\mathbb{M}_2[1])$ and $\mathbf{DA}(S)(\mathcal{R}\mathbb{M}_1, \mathcal{R}\mathbb{M}_2[2])$. One can show that they do not always coincide with the Yoneda Ext-groups in the exact category $\mathcal{M}_1(S)$; for instance for $S = \mathbb{P}^1_\mathbb{C}$, $\mathcal{M}_1 = [\mathbb{Z} \to 0]$ and $\mathcal{M}_2 = [0 \to \mathbb{G}_\mathrm{m}]$, the second Yoneda Ext group $\mathrm{YExt}^2_{\mathcal{M}_1(S)}([\mathbb{Z} \to 0], [0 \to \mathbb{G}_\mathrm{m}]) = 0$ (this follows from arguments of [Org04, Proposition 3.2.4] together with the fact that any abelian scheme over $\mathbb{P}^1$ comes from an abelian variety over $\mathbb{C}$) while

$$\mathbf{DA}(\mathbb{P}^1)(\mathcal{R}[\mathbb{Z} \to 0], \mathcal{R}[0 \to \mathbb{G}_\mathrm{m}][2]) \simeq \mathbf{DA}(\mathbb{P}^1)(\mathbb{Q}, \Sigma^\infty\mathbb{G}_\mathrm{m} \otimes \mathbb{Q}[1]) \simeq \mathbb{Q}.$$

3.5 Conjectures

In this final section, we state natural conjectures on the topics of this chapter, some of which we believe are accessible using the methods of this thesis, but could not be fully explored in the scope of this work. To make the statements more concise, we only state them for cohomological 1-motives; the conjectures not involving ω^1 can be transposed to $\mathbf{DA}_1(S)$.

First, we assert that the t-structure behaves well with respect to compactness.

Conjecture 3.5.1. *Let S be a noetherian finite-dimensional scheme. The t-structure $t^1_{\mathbf{MM}}$ on $\mathbf{DA}^1(S)$ restricts to a t-structure on the subcategory $\mathbf{DA}^1_c(S)$. Equivalently, the truncation functor $\tau_{\geq 0} : \mathbf{DA}^1(S) \to \mathbf{DA}^1(S)_{\geq 0}$ preserves compact objects. Moreover, if we denote by $\mathbf{MM}^1_c(S)$ the heart of the resulting t-structure, we have that $\mathbf{MM}^1(S) \simeq \operatorname{Ind}\mathbf{MM}^1_c(S)$.*

Assuming this conjecture, there should be a closer relationship between $\mathcal{DG}_S$ and $\mathbf{MM}^1_c(S)$, as follows.

Conjecture 3.5.2. *Let S be a noetherian finite-dimensional scheme.*

- *We have* $\mathbf{DA}^1_c(S)_{\leq 0} = \langle \mathcal{DG}_S(-1)\rangle_-$.
- *The category* $\mathbf{MM}^1_c(S)$ *is generated as an abelian category by* $\mathcal{DG}_S(-1)$.
- *For any* $M \in \mathbf{MM}^1_c(S)$, *there exists a locally closed stratification* (S_i) *of* S *such that* $(S_i \to S)^*M$ *lies in* $\mathcal{RM}_1(S_i)$ *for all* i.

Here are some exactness properties which seem to lie deeper than those of Proposition 3.4.8.

Conjecture 3.5.3. *Let* $f : S \to T$ *be a morphism between noetherian, finite dimensional schemes.*

- *The functor* $f^* : \mathbf{DA}^1(T) \to \mathbf{DA}^1(S)$ *is t-exact.*
- *Assume that* f *is proper of relative dimension* d. *Then the functor* $\omega^1 f_* : \mathbf{DA}^1(S) \to \mathbf{DA}^1(T)$ *has cohomological amplitude* $[-d-1, 0]$.

Assuming the previous conjecture on pullbacks, it makes sense to state the following.

Conjecture 3.5.4. *For any closed immersion* $i : Z \to S$ *and complementary open immersion* $j : U \to S$, *the t-structure* $t^1_{\mathbf{MM}}(S)$ *is obtained by gluing the t-structures* $t^1_{\mathbf{MM}}(U)$ *and* $t^1_{\mathbf{MM}}(Z)$ *using the functors* $i^* \dashv i_* \dashv \omega^1 i^!$ *and* $j_! \dashv j^* \dashv \omega^1 j_*$ *in the sense of [BBD82, 1.4].*

Conjecture 3.5.5. *Let* S *be a noetherian scheme of dimension* $\leq d$. *Then for all* $M, N \in \mathcal{DG}_S$ *and* $n \geq d+2$, *we have*

$$\mathbf{DA}(S)(M, N[n]) = 0.$$

In particular, by [Ayo07a, Proposition 2.1.73], the t-structure $t^1_{\mathbf{MM}}$ *is non-degenerate.*

We turn to the question of realization functors, which have not been considered in this chapter.

Conjecture 3.5.6. *The standard realisation functors on* $\mathbf{DA}_1(S)$ *are t-exact and conservative. More precisely, the following holds.*

- *Let* S *be a finite type scheme over* $\mathbf{Spec}(k)$ *with* k *a field of characteristic* 0 *with a fixed embedding* $\sigma : k \to \mathbb{C}$. *Consider the Betti realisation functor*

  $$R_{B,\sigma} : \mathbf{DA}_1(S) \longrightarrow D(S^{\mathrm{an}})$$

  with target the derived category of sheaves of $\mathbb{Q}$*-vector spaces on the analytic space* $S^{\mathrm{an}} = (S \times_{k,\sigma} \mathbb{C})^{\mathrm{an}}$, *endowed with its standard t-structure [Ayo10b, Definition 2.1].*

  Then $R_{B,\sigma}$ *is t-exact and conservative.*

- *Let* ℓ *be a prime and* S *be a noetherian finite dimensional excellent scheme over* $\mathbf{Spec}(\mathbb{Z}[1/\ell])$. *Consider the* ℓ*-adic realisation functor*

  $$R_\ell : \mathbf{DA}_{1,c}(S) \longrightarrow \hat{D}^{\text{ét}}_{ct}(S, \mathbb{Q}_l)$$

  with target a version of the constructible derived category of $\mathbb{Q}_l$*-adic sheaves [Ayo14a, Definition 9.3, Proposition 9.5]. This category is the pseudo-abelian completion of a full t-subcategory of the derived category with the standard t-structure and inherits a "standard" t-structure in this way.*

  Then R_ℓ *is t-exact and conservative.*

In characteristic 0, we expect a tighter relationship with Deligne 1-motives, in the following way.

Conjecture 3.5.7. *Let* S *be a noetherian finite dimensional normal scheme of characteristic 0. Then smooth homological* 1*-motives are geometrically smooth, i.e.,* $\mathbf{DA}_{1,\mathrm{gsm}}(S) = \mathbf{DA}_{1,\mathrm{sm}}(S)$. *Moreover, the t-structure on* $\mathbf{DA}_1(S)$ *restricts to a t-structure on* $\mathbf{DA}_{1,\mathrm{sm}}(S)$, *and the resulting t-category is equivalent to the derived category* $D(\mathrm{Ind}(\mathcal{M}_1(S))$ *of the abelian category of Ind-Deligne* 1*-motives over* S.

Remark 3.5.8. It is possible to prove, starting from Grothendieck's theorem on extensions of abelian schemes in characteristic 0 [Gro66a, Corollaire 4.5], that the category $\mathcal{M}_1(S)$ is abelian for S normal noetherian of characteristic 0 (so that the last part of the conjecture makes sense). We do not know the status of this result in greater generality, and do not know what the category of smooth 1-motives should look like in general.

Finally, here is a (admittedly loosely stated) conjecture on the special situation where S is a trait.

Conjecture 3.5.9. *Let $S = \mathbf{Spec}(R)$ be the spectrum of a discrete valuation ring, with generic point η and special point s.*

(i) *Let π be a uniformizer of R. The motivic nearby cycle $\Psi_\pi : \mathbf{DA}_1(\eta) \to \mathbf{DA}_1(s)$ is t-exact, the induced functor $\Psi_\pi : \mathbf{MM}_{1,c}(\eta) \to \mathbf{MM}_{1,c}(s)$ and its monodromy [Ayo14a, §11] can be computed in terms of Deligne 1-motives and the result matches the constructions of [Ray94].*

(ii) *Let A be an abelian variety and $N^\circ(A)$ its connected Néron model. Then*

$$\omega^{\leq 1} j_*(\Sigma^i nfty A_{\mathbb{Q}}(-1)) \simeq \Sigma^\infty N^\circ(A)_{\mathbb{Q}}(-1)$$

(iii) *Generalizing (ii), there is a the relationship between the functor ω^1_S and the theorems of Raynaud [Ray70a] and Pépin [Pép13] on the specialisation of Picard functors.*

Finally, one should consider extensions of the results of this chapter and of the conjectures below to motives with "integral" coefficients (more precisally, in rings Λ such that any prime number is invertible either in Λ or on the base scheme) in the spirit of [BVK10]. More precisely, it is expected that the motivic t-structure on $\mathbf{DA}^{\text{ét}}(S, \mathbb{Q})$ extends to a suitably defined triangulated category of étale motives with integral coefficients (but not to the triangulated category of Nisnevich motives with integral coefficients). The book [BVK10] presents a theory of 1-motives over a perfect field k of exponential characteristic p in the category $\mathbf{DM}^{\text{ét}}_c(k, \mathbb{Z}[1/p])$ of constructible étale motives with transfers with coefficients in $\mathbb{Z}[1/p]$. Two key ingredients are a generalisation of Deligne 1-motives to allow for torsion in the weight 0 lattice and the rigidity theorem of Suslin-Voevodsky which describes étale motives with finite prime-to-p coefficients. These ingredients are still available for $\mathbf{DA}(-)$ over a general base if we invert all residual characteristics of S (via the relative rigidity theorem of [Ayo14a, Theoreme 4.1]).

Appendix

3.A Deligne 1-motives

We gather necessary results on Deligne 1-motives [Del74, §10] over general base schemes which we could not find in the literature. Useful references besides Deligne's original work are [Jos09], [BVK10, Appendix C].

3.A.1 Definitions

Definition 3.A.1. Let S be a scheme. We say that a group scheme G/S is

(i) *discrete* if it is étale locally constant finitely generated.

(ii) a *lattice* if it is discrete and torsion free.

(iii) a *abelian-by-torus scheme* if it is semi-abelian of locally constant toric rank (hence a uniquely defined extension of a torus by an abelian scheme by [FC90, 2.11]).

Definition 3.A.2. Let S be a scheme. A 2-term complex of commutative S-group schemes:

$$M = [\overset{0}{L} \longrightarrow \overset{-1}{G}]$$

is called a *Deligne* 1*-motive* over S if L is a lattice and G is a abelian-by-torus scheme. A morphism of Deligne 1-motives is a morphism of complexes of group schemes, or equivalently a morphism of complex of representable sheaves on $(\mathsf{Sm}/S)_{\text{ét}}$. We denote by $\mathcal{M}_1(S,\mathbb{Z})$ the category of Deligne 1-motives. It is a pseudo-abelian additive category, with biproducts induced by fiber products of S-group schemes.

A Deligne 1-motive $M = [L \to G]$ comes with a 3-term functorial weight filtration, defined as follows.

$$W_{-2}M = [0 \longrightarrow T]$$
$$W_{-1}M = [0 \longrightarrow G]$$
$$W_0 M = M$$

Notation 3.A.3. Let $f : [L \to G] \to [L' \to G']$ be a morphism of Deligne 1-motives. We use the notation f_L, f_G, f_A, f_T for the induced maps $\mathrm{Gr}_0^W f : L \to L'$, $W_0 f : G \to G'$, $\mathrm{Gr}_1^W f : A \to A'$, $\mathrm{Gr}_2^W f : T \to T'$.

We have a basic contravariant fonctoriality:

Definition 3.A.4. Let $f : S' \to S$ be any morphism of schemes. Then pullback of S-group schemes along f induces an additive functor:

$$f^* : \mathcal{M}_1(S,\mathbb{Z}) \to \mathcal{M}_1(S',\mathbb{Z})$$

We are not so much interested in 1-motives per se as in the objects they define in the derived category of sheaves with rational coefficients.

Lemma 3.A.5. *Any morphism in $\mathcal{M}_1(S,\mathbb{Z})$ which induces a quasi-isomorphism of complexes of abelian sheaves on $(\mathsf{Sm}/S)_{\text{ét}}$ is an isomorphism.*

Proof. Let $f = (f_L, f_G) : [L_1 \longrightarrow G_1] \to [L_2 \longrightarrow G_2]$ be a quasi-isomorphism of complexes of sheaves. By a diagram chase, this is equivalent to $\mathrm{Ker}(f_L) \simeq \mathrm{Ker}(f_G)$ and $\mathrm{Coker}(f_L) \simeq \mathrm{Coker}(f_G)$. Since $\mathrm{Ker}(f_L)$ is locally constant finitely generated free and $\mathrm{Ker}(f_G)$ is a group scheme whose identity component is semi-abelian and with finite π_0, they must be both trivial. Similarly, $\mathrm{Coker}(f_L)$ is discrete and $\mathrm{Coker}(f_G)$ has connected fibers, so they must be both trivial. Hence f is an isomorphism. □

We can consequently think of $\mathcal{M}_1(S, \mathbb{Z})$ as a full subcategory of $D(\mathbf{Cpl}(\mathbf{Sh}((Sm/S)_{\text{ét}}, \mathbb{Z})))$.

Definition 3.A.6. Let S be a noetherian scheme. We write $\mathcal{M}_1(S)$ for the category $\mathcal{M}_1(S, \mathbb{Z}) \otimes \mathbb{Q}$. We say that a morphism in $\mathcal{M}_1(S)$ is integral if it comes from $\mathcal{M}_1(S, \mathbb{Z})$. For $f : S' \to S$ morphism of schemes, we still write f^* for the induced additive functor $\mathcal{M}(S) \to \mathcal{M}(S')$.

By the results above, we can and do think of $\mathcal{M}_1(S)$ as a full subcategory of $D(\mathbf{Cpl}(\mathbf{Sh}((\mathsf{Sm}/S)_{\text{ét}}, \mathbb{Q})))$.

3.A.2 Continuity and smoothness

We think of Deligne 1-motives as "motivic local systems" over the base S. The results in this section, which have classical analogues for local systems/lisse sheaves, justify in part this intuition.

We start with a lemma about discrete group schemes:

Lemma 3.A.7. *Let S be a locally noetherian japanese scheme, η its scheme of generic points. Then the category of discrete group schemes on η is the 2-colimit of the categories of discrete group schemes on dense open subschemes of S. The same statement holds for the category of lattices.*

Proof. The statement is equivalent to the following.

(i) For L/η discrete group scheme, there exists $U \subset S$ dense open and L'/U discrete such that $L \simeq \eta^* L'$. Moreover, if L is a lattice, one can choose L' be a lattice as well.

(ii) For $U \subset S$ dense open, $L, L'/U$ discrete, we have

$$\mathrm{Hom}(\eta^* L, \eta^* L') \simeq \mathrm{Colim}_{V \subset U} \mathrm{Hom}((V \to U)^* L, (V \to U)^* L').$$

By the topologically invariance of the étale site, we can assume S to be reduced. Since S is locally noetherian japanese and reduced, the normal locus of S is open [Gro65, Proposition 6.13.2]. So any small enough open set U in S is normal, and in particular geometrically unibranch. By the discussion in [SGA70, Exp. X 6.2], discrete group schemes on geometrically unibranch schemes are split by finite étale covers. Moreover, for any small enough open set U the set of connected components (open by local noetherianness) of U and of η coincide. We can thus reduce to the case where η is connected (i.e., S irreducible).

We prove (i). Since η itself is normal, there is a finite étale Galois cover $\tilde{\eta}/\eta$ such that $L_{\tilde{\eta}}$ is constant. In other words, L corresponds to a representation ρ of $\mathrm{Gal}(\tilde{\eta}/\eta)$ on a finitely generated abelian group F. By [Gro66b, Théorème 8.8.2, Théorème 8.10.5] and [Gro67, Théorème 17.7.8] there exists a $U \subset S$ dense open and $\tilde{U}/U$ finite étale such that $\tilde{U} \times_U \eta \simeq \tilde{\eta}$. Up to shrinking U, one can assume it to be normal. By [Gro66b, Théorème 8.8.2] applied to the finite group $\mathrm{Gal}(\tilde{\eta}/\eta)$, up to shrinking U one can assume that $\mathrm{Aut}(\tilde{U}/U) \simeq \mathrm{Gal}(\tilde{\eta}/\eta)$ (in particular $\tilde{U}/U$ is Galois). Then the representation of $\mathrm{Gal}(\tilde{U}/U)$ on F corresponding to ρ via this isomorphism defines a discrete group scheme L'/U such that $L \simeq \eta^* L'$ as required. The addendum about lattices follows from the construction.

We now prove (ii). Let $U \subset S$ dense open, $L, L'/U$ discrete group schemes. We can shrink U and assume it is normal. Let $\tilde{V}/V$ be a finite étale Galois covering trivializing L and L'. We thus get two finitely generated abelian groups F, F' with representations ρ, ρ' of $\mathrm{Gal}(\tilde{V}/V)$. Let $\tilde{\eta} := \tilde{V} \times_V \eta$. Then $\tilde{\eta}/\eta$ is Galois with $G := \mathrm{Gal}(\tilde{V}/V) \simeq \mathrm{Gal}(\tilde{\eta}/\eta)$. Then the system in the right-hand side of (ii) is constant and both sides of (ii) are in bijection with $\mathrm{Hom}_G(\rho, \rho')$. This concludes the proof. □

Remark 3.A.8. It is not clear to the author how to extend this result to a more general continuity result for discrete group schemes on a projective limit of schemes with affine transition morphisms.

We deduce from this a continuity result for Deligne 1-motives.

Proposition 3.A.9. *Let S be a locally noetherian japanese scheme, η its scheme of generic points. Then the category $\mathcal{M}_1(\eta,\mathbb{Z})$ (resp. $\mathcal{M}_1(\eta)$) is the 2-colimit of the categories $\mathcal{M}_1(U,\mathbb{Z})$ (resp. $\mathcal{M}_1(U)$) for all dense opens $U \subset S$.*

Proof. The case of $\mathcal{M}_1(-)$ follows immediately from the one of $\mathcal{M}_1(-,\mathbb{Z})$. We have to show that

(i) for all $M \in \mathcal{M}_1(\eta,\mathbb{Z})$, there exists $U \subset S$ dense open and $M' \in \mathcal{M}_1(U,\mathbb{Z})$ such that $M \simeq \eta^* M'$, and that

(ii) for all $U \subset S$ dense open and all $M, N \in \mathcal{M}_1(U,\mathbb{Z})$:

$$\mathcal{M}_1(\eta,\mathbb{Z})(\eta^* M, \eta^* N) \simeq \mathrm{Colim}_{V\subset U}\mathcal{M}_1(V,\mathbb{Z})((V\to U)^* M, (V\to U)^* N).$$

We prove (i). Let $M = [L \to G] \in \mathcal{M}_1(\eta,\mathbb{Z})$ with the extension $0 \to T \to G \to A \to 0$.

By [Gro66b, Théorème 8.8.2.(ii), Scholie 8.8.3, Théorème 8.10.5.(xii)] and [Gro67, Proposition 17.7.8], we can find an $U \subset S$ and a smooth group scheme G'/U such that $G \simeq G' \times_U \eta$. Recall that an abelian scheme is by definition a smooth proper group scheme with connected fibers, hence by [Gro66b, Théorème 8.8.2.(ii), Scholie 8.8.3, Théorème 8.10.5.(xii)] and [Gro67, Proposition 17.7.8], we can shrink U and find an abelian scheme A'/U such that $A \simeq A' \times_U \eta$. By Lemma 3.A.7 and the duality between lattices and tori, we can shrink U and assume that there exists a discrete group scheme L' and a torus T' over U such that $L \simeq L' \times_U \eta$ and $T \simeq T' \times_U \eta$.

We have spread out the pure pieces of M, it remains to glue them together. By [Gro66b, Théorème 8.8.2.(i)], up to shrinking U, we have morphisms $A' \to G' \to T'$ which restrict to the extension defining G. By a standard argument based on [Gro66b, Théorème 8.10.5], up to shrinking U, this is in fact an exact sequence of group schemes. Finally, we have to spread out the morphism $L \to G$. This can be done by the same Galois descent argument as in the end of the proof of Lemma 3.A.7.

Let us now prove (ii). In $\mathcal{M}_1(-,\mathbb{Z})$, the components of a morphism are morphisms of (group) schemes. It is enough to spread them out one by one because the resulting diagram will comute by density. We have treated morphisms of discrete group schemes in Lemma 3.A.7. The case of morphisms of semi-abelian schemes (which are in particular of finite presentation) is a direct application of [Gro66b, Théorème 8.8.2.(i)]. □

When the base scheme is noetherian excellent and reduced (resp. normal), we can say more.

Proposition 3.A.10. *Let S be a noetherian excellent scheme, $i : \eta \to S$ its scheme of generic points.*

(i) *Suppose S reduced. Then the pullback functor $\eta^* : \mathcal{M}_1(S,\mathbb{Z}) \to \mathcal{M}_1(\eta,\mathbb{Z})$ (resp. $\eta^* : \mathcal{M}_1(S) \to \mathcal{M}_1(\eta)$) is faithful.*

(ii) *Suppose moreover that S is normal. Then η^* is fully faithful.*

Proof. Let us prove (i). By Proposition 3.A.9 this is equivalent to the faithfulness of the functor j^* for all $j : U \to V$ dense open subsets. It is enough to show faithfullness of j^* separately for morphisms of discrete group schemes and semi-abelian schemes, and in both discrete group schemes and semi-abelian schemes it follows from the "reduced to separated" uniqueness criterion [Gro60, Lemme 7.2.2.1].

We now prove (ii). By Proposition 3.A.9, it is enough to prove fullness for the functor j^* for all $j : U \to V$ dense opens. Let $M = [L \xrightarrow{u} G]$, $M' = [L' \xrightarrow{u'} G'] \in \mathcal{M}_1(V,\mathbb{Z})$ and $f_U = (f_U^L, f_U^G) : j^* M \to j^* M'$. First, using the normality of V and [SGA03, Exposé I Corollaire 10.3], the morphism f_U^L extends uniquely to a morphism $f^L : L \to L'$. Second, using the normality of V and [FC90, Proposition 2.7], the morphism f_U^G extends uniquely to a morphism $f^G : G \to G'$. The uniqueness ensures that (f^L, f^G) is a morphism $M \to M'$ which extends f_U. □

3.A.3 Pushforward and Weil restriction

Let $g : S' \to S$ be a finite étale morphism. We can define a pushforward functor $g_* : \mathcal{M}_1(S') \to \mathcal{M}_1(S)$ using Weil restriction of scalars. We also want to study the problem of pushforward for more general morphisms. Recall the following definition:

Definition 3.A.11. Let $g : S' \to S$ be a morphism of schemes and X/S' be a S'-scheme. The *Weil restriction* $R_g X$ is the presheaf of sets on Sch/S defined for any S-scheme U by:

$$R_g X(U) = X(U \times_S S')$$

If X/S' is a commutative group scheme (or more generally an fppf sheaf of abelian groups on Sch/S), then $R_g X$ is naturally an fppf sheaf of abelian groups on Sch/S. Moreover, the formation of R_g is functorial and compatible with base change. We summarize results from the litterature.

Proposition 3.A.12. *Let $g : S' \to S$ be a morphism of schemes and X/S' be a S'-scheme.*

(i) *[Ols06b, Theorem 1.5] Assume that g is proper flat of finite presentation. Then $R_g X$ is representable by an algebraic space (Note that we will only need the case g finite flat, which is presumably much easier, but I could not find a reference).*

(ii) *[BLR90b, 7.6/5] Assume that g is finite flat. Then if X is smooth (resp. of finite presentation) then $R_g X$ (which exists at least as an algebraic space by (i)) is smooth (resp. of finite presentation).*

(iii) *[BLR90b, 7.6/5] Assume that g is finite étale. Then if X is proper then $R_g X$ (which exists at least as an algebraic space by (i)) is proper.*

(iv) *[BLR90b, 7.6/2] Let $h : X \to Y$ be a closed immersion of S'-schemes. Then $R_g h : R_g X \to R_g Y$ is a closed immersion of presheaves. As a corollary, if X/S if affine, then $R_g X$ is representable by an affine scheme.*

We now use the results above to analyse Weil restriction of pure 1-motives. We are spared from having to consider algebraic spaces by the following result.

Proposition 3.A.13. *Let $g : S' \to S$ be finite flat.*

1. *Let T/S' be a torus (resp. L/S' be a lattice). Then $R_g T$ is a torus (resp. $R_g L$ is a lattice).*
2. *Let A/S' be an abelian scheme. Assume that g is étale. Then $R_g A$ is an abelian scheme.*

Proof. By Proposition 3.A.12 (iv), we know that $R_g T$ and $R_g L$ are representable by affine S'-group schemes. Moreover, because of the compatibility with base change, it is enough to consider the case of a split torus and a constant lattice over S', in which case the Weil restrictions are directly seen to be a split torus or a constant lattice over S.

By Proposition 3.A.12 (i)-(iii), we know that $R_g A$ is representable by a proper smooth algebraic group space over S. By [FC90, Theorem 1.9], this implies that $R_g A$ is an abelian scheme. □

Now we tackle the case of semi-abelian schemes.

Lemma 3.A.14. *Let $g : S' \to S$ be a morphism of schemes.*

(i) *When restricted to fppf sheaves of abelian groups, the functor R_g is left exact.*

(ii) *Assume that g is finite flat. Let $f : G \to H$ be a smooth and surjective morphism between commutative groups schemes of finite presentation. Then the morphism of algebraic group spaces $R_g f : R_g G \to R_g H$ is smooth and surjective.*

(iii) *Assume g is finite flat. Let $0 \to G' \xrightarrow{i} G \xrightarrow{p} G'' \to 0$ be an exact sequence of smooth commutative S-group schemes with $G \to G''$ flat (and hence smooth). The sequence $0 \to R_g G' \to R_g G \to R_g G'' \to 0$ is exact.*

Proof. Point (i) is clear from the definition. We turn to point (ii). The fact that $R_g f$ is smooth follows from the infinitesimal criterion of smoothness (and does not require that we are working with group schemes). The surjectivity can be tested point-wise on S, so that by compability of R_g with base change we can assume that S is the spectrum of a field k. Surjectivity is a geometric property, so that we can assume k to be algebraically closed as well. We then have to check the surjectivity of the induced map $R_g G(k) = G(S') \to R_g H(k) = H(S')$ on k-points. Since S'/k is finite flat, it is a product of finite local algebras. Surjectivity then follows from the surjectivity of f, the fact that k is algebraically closed, and the formal smoothness of f. Note that if g is finite étale, we do not need f smooth.

For (iii), by the remarks before the statement, it is enough to check that $R_g G'$ is the scheme-theoretic kernel of $R_g p$ and that $R_g p$ is an fppf morphism. The first assertion follows from (i), and the second from (ii). □

Proposition 3.A.15. *Let $g : S' \to S$ be finite étale and G/S be an abelian-by-torus scheme. Then $R_g G$ is an abelian-by-torus scheme.*

Proof. The result follows directly from Proposition 3.A.13 and Lemma 3.A.14 (iii). □

Définition 3.A.16. Let $g : S' \to S$ be a finite étale morphism. We define the Weil restriction of a Deligne 1-motive $M = [L \overset{u}{\to} G] \in \mathcal{M}_1^{\mathbb{Z}}(S')$ as $R_g M = [R_g L \overset{R_g u}{\to} R_g G]$ which is in $\mathcal{M}_1(S)$ by Propositions 3.A.13 and 3.A.15. This induces a functor

$$g_* : \mathcal{M}_1^{\mathbb{Z}}(S') \to \mathcal{M}_1^{\mathbb{Z}}(S)$$

which is by construction a right adjoint to g^*.

3.B Motivic cohomology in degrees $(*, \leq 1)$

We gather here some computations of motivic cohomology groups which are necessary for the proof of the main results. Many of these results are directly taken from [Ayo14a, §11]. We deduce the others from simple adaptations of arguments from loc. cit. and from the proof of Proposition 3.1.11, except that we introduce explicit maps computing certain groups.

Notation 3.B.1. For $p, q \in \mathbb{Z}$, we write $H^{p,q}_{\mathcal{M}}(S) := \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S(q)[p])$.

Proposition 3.B.2. *[Ayo14a, Proposition 11.1 (b)] Motivic cohomology of negative weights vanishes: for all $w < 0$ and $n \in \mathbb{Z}$, we have $H^{n,w}_{\mathcal{M}}(S) \simeq 0$.*

Let S be a scheme. Then we have $D(\mathsf{Sm}/S)(\mathbb{Q}_S, \mathbb{Q}_S) \simeq \mathbb{Q}^{\pi_0(S)}$ (with $\pi_0(S)$ the set of connected components of S). This provides a morphism

$$\nu^{0,0} : \mathbb{Q}^{\pi_0(S)} \simeq D(\mathsf{Sm}/S)(\mathbb{Q}_S, \mathbb{Q}_S) \longrightarrow \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S) = H^{0,0}_{\mathcal{M}}(S)$$

More generally, we have for all $n \in \mathbb{Z}$ a morphism

$$\nu^{n,0} : D(\mathsf{Sm}/S)(\mathbb{Q}_S, \mathbb{Q}_S[n]) \longrightarrow H^{n,0}_{\mathcal{M}}(S)$$

Proposition 3.B.3.

(i) *For all $n < 0$, we have $H^{n,0}_{\mathcal{M}}(S) \simeq 0$.*

(ii) *The morphism $\nu^{0,0}$ induces an isomorphism $H^{0,0}_{\mathcal{M}}(S) \simeq \mathbb{Q}^{\pi_0(S)}$.*

(iii) *Assume S regular. For all $n > 0$, we have $H^{n,0}_{\mathcal{M}}(S) \simeq 0$.*

(iv) *Let $f : T \to S$ be a smooth surjective morphism with geometrically connected generic fibers. Then for all $n \in \mathbb{Z}$, we have $f^* : H^{n,0}_{\mathcal{M}}(S) \xrightarrow{\sim} H^{n,0}_{\mathcal{M}}(T)$.*

Proof. Statements (i) and (ii) are proved in [Ayo14a, Proposition 11.1 (a)]. More precisely, in [Ayo14a, Proposition 11.1 (a)], an unspecified isomorphism $H^{0,0}_{\mathcal{M}}(S) \simeq \mathbb{Q}^{\pi_0(S)}$ is constructed. Let us sketch why it corresponds to $\nu^{0,0}$ by going through the proof in loc. cit.

Step A is a reduction to rational coefficients and is irrelevant to us.

Step B consists of several reductions. First we have a reduction to the affine case via a Mayer-Vietoris sequence, and reduction to S of finite type over a Dedekind scheme D by a filtered projective limit argument. The Mayer-Vietoris sequence for the Zariski topology is also available for $D(\mathsf{Sm}/S)$ and is compatible with the one for $H^{n,0}_{\mathcal{M}}$ via the $\nu^{n,0}$. From [Gro66b, Proposition 8.4.2], we see that $\pi_0(S)$ is compatible with filtered projective limits. Then there is an induction on the dimension. The case $\dim(S) = 0$ uses the comparison with $\mathbf{DM}^{\text{ét}}(k)$, the cancellation theorem to reduce to a computation in the derived category of sheaves with transfers on SmCor/k. To see that this computation is compatible with the map $\nu^{0,0}$, one just needs to introduce the similar map on sheaves with transferts and look at the equivalence $\mathbf{DA}(k) \simeq \mathbf{DM}(k)$. The last part of Step B is a reduction to the normalisation. It relies on localisation and base change for finite morphisms, both of which are available for $D(\mathsf{Sm}/S)$, and compatible with the $\nu^{n,0}$ maps.

Step C settles the case of a normal positive characteristic scheme. The argument reduces to the case of a smooth $\mathbb{F}_p$ scheme via de Jong's alterations. The main problem for the alteration argument is that the proper base change theorem does not hold in general for $D(\mathsf{Sm}/S)$, and so we do not have a priori the analogue of the long exact sequence (111) of loc. cit. However, one can first prove that $\nu^{n,0}$ is an isomorphism in the regular case, and then deduce that in the case of interest, the proper base change map is an isomorphism. We leave the details to the reader. For the last part of step C, namely the case of a smooth $\mathbb{F}_p$-variety X, we have $H^0_{\text{ét}}(X, \mathbb{Q}) = \mathbb{Q}^{\pi}_0(X)$ and $\nu^{n,0}$ is clearly an isomorphism in this case.

Step D settles the case where the fiber in characteristic 0 is non-empty. The resolution and alteration arguments have to be adapted as in step C. We are then reduced to the case S smooth over D. We have $H^{n,0}_{\mathcal{M}}(S, \mathbb{Q}) \simeq H^{n,0}_{\mathcal{M}}(S \times \mathbb{Q}, \mathbb{Q})$ by absolute purity, which is not available in $D(\mathsf{Sm}/S)$. To remedy this, we have to show that $H^0_{\text{ét}}(S, \mathbb{Q}) \simeq H^0_{\text{ét}}(S \times \mathbb{Q}, \mathbb{Q})$. This follows from the fact that since S/D is smooth, we have $\pi_0(S) \simeq \pi_0(S \times \mathbb{Q})$.

This completes the sketch of the proof that $\nu^{0,0}$ is an isomorphism.

Let us prove Statement (iii). Fix $n > 0$. We can assume that S is connected with generic point η. By the argument at the beginning of the proof of [Ayo14a, Corollaire 11.4], combining absolute purity and localisation with the vanishing of negative motivic cohomology 3.B.2, one can deduce that for any dense open set U in S, the restriction map $H^{n,0}_{\mathcal{M}}(S) \to H^{n,0}_{\mathcal{M}}(U)$ is injective. By the continuity property of [Ayo14a, Proposition 3.20], we deduce that the restriction map $H^{n,0}_{\mathcal{M}}(S) \to H^{n,0}_{\mathcal{M}}(\eta)$ is injective. So we are reduced to the case where S is the spectrum of a field k.

By separation, we can assume that k is perfect. By [CDb, Corollary 16.2.22], we reduce to compute $\mathbf{DM}(k, \mathbb{Q})(\mathbb{Q}_k, \mathbb{Q}_k[n])$. By the cancellation theorem [Voe10], we reduce to compute $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})(\mathbb{Q}_k, \mathbb{Q}_k[n])$. Since the sheaf $\mathbb{Q}_k$ is both cofibrant and $\mathbb{A}^1$-local, this coincides with the same Hom group computed in the derived category of étale sheaves over Sm/S, which vanishes. This concludes the proof of (iii).

Let us prove Statement (iv). By Mayer-Vietoris, we can assume S to be affine. By a limit argument using the continuity property of $\mathbf{DA}$, we can then assume that S is of finite type over a Dedekind ring. Using [dJ97, Corollary 5.15] applied to the irreducible components of the normalisation of S and then iterating, we build a proper hypercovering $\pi_\bullet : \widetilde{S}_\bullet \to S$ with all $\widetilde{S}_n$ regular. We pullback $\pi_\bullet$ to obtain a proper hypercovering $\pi'_\bullet : \widetilde{T}_\bullet \to T$. Since f is smooth, all $\widetilde{T}_n$ are regular as well. By cohomological descent for the h-topology [CDb, Theorem 14.3.4], we have $\mathbb{Q}_S \simeq \pi_{\bullet *}\mathbb{Q}_{\widetilde{S}_\bullet}$ and $\mathbb{Q}_T \simeq \pi'_{\bullet *}\mathbb{Q}_{\widetilde{T}_\bullet}$. We deduce that $H^{n,0}_{\mathcal{M}}(S) \simeq \mathbf{DA}(\widetilde{S}_\bullet)(\mathbb{Q}_{\widetilde{S}_\bullet}, \mathbb{Q}_{\widetilde{S}_\bullet}[n])$ and $H^{n,0}_{\mathcal{M}}(T) \simeq \mathbf{DA}(\widetilde{T}_\bullet)(\mathbb{Q}_{\widetilde{T}_\bullet}, \mathbb{Q}_{\widetilde{T}_\bullet}[n])$. . By (i), (ii) and (iii), we have for every $k, m \in \mathbb{Z}$ that $\mathbf{DA}(\widetilde{S}_k)(\mathbb{Q}_{\widetilde{S}_k}, \mathbb{Q}_{\widetilde{S}_k}[m])$ is isomorphic to $BQ^{\pi_0(\widetilde{S}_k)}$ if $m = 0$ and 0 otherwise; a similar formula holds for $\widetilde{T}$. Now the map f and its pullbacks induce isomorphisms $\pi_0(S_k) \simeq \pi_0(T_k)$ on sets of connected components because f has geometrically connected generic fibers (a property which is itself stable by pullback). This implies the result. $\square$

Let S be a scheme. We have $D(\mathsf{Sm}/S)(\mathbb{Q}_S, \mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}) \simeq H^0(S_{\text{ét}}, \mathbb{G}_{\mathrm{m}} \otimes \mathbb{Q}) \simeq \mathcal{O}^\times(S) \otimes \mathbb{Q}$ and

$D(\mathsf{Sm}/S)(\mathbb{Q}_S, \mathbb{G}_m \otimes \mathbb{Q}[1]) \simeq H^1(S_{\text{ét}}, \mathbb{G}_m \otimes \mathbb{Q}) \simeq \mathrm{Pic}(S) \otimes \mathbb{Q}$. Combining these isomorphisms with Proposition 3.2.9, this induces morphisms

$$\nu^{1,1} : \mathcal{O}^\times(S) \longrightarrow H^{1,1}_{\mathcal{M}}(S)$$

and

$$\nu^{2,1} : \mathrm{Pic}(S)_{\mathbb{Q}} \longrightarrow H^{2,1}_{\mathcal{M}}(S).$$

More generally, for any $n \in \mathbb{Z}$, we have an induced morphism

$$\nu^{n,1} : D(\mathsf{Sm}/S)(\mathbb{Q}_S, \mathbb{G}_m[n-1]) \to H^{n,1}_{\mathcal{M}}(S).$$

Proposition 3.B.4.

(i) *For all $n \leq 0$, we have $H^{n,1}_{\mathcal{M}}(S) \simeq 0$.*

(ii) *Assume S regular. The morphism $\nu^{1,1}$ induces an isomorphism $H^{1,1}_{\mathcal{M}}(S) \simeq \mathcal{O}^\times(S)_{\mathbb{Q}}$.*

(iii) *Assume S regular. The morphism $\nu^{2,1}$ induces an isomorphism $H^{2,1}_{\mathcal{M}}(S) \simeq \mathrm{Pic}(S)_{\mathbb{Q}}$.*

(iv) *Assume S regular. For all $n \neq 1, 2$, we have $H^{n,1}_{\mathcal{M}}(S) \simeq 0$. We have also $D(\mathsf{Sm}/S)(\mathbb{Q}_S, \mathbb{G}_m \otimes \mathbb{Q}[n-1]) \simeq 0$, so that the morphism $\nu^{n,1}$ is an isomorphism.*

Proof. Statement (i) for S regular and a weaker version of (ii) (without specifying the isomorphism) are proved in [Ayo14a, Corollaire 11.4].

To pass from (i) for S regular to a general S, we apply resolution of singularities by alterations and cohomological h-descent for a proper regular hypercovering (which induces a descent spectral sequence for $H^{n,1}(-)$). To be more precise, one has to reduce to a situation where one can apply De Jong's theorem, e.g. S of finite type over a Dedekind ring: for this, one uses Mayer-Vietoris to first reduce to S affine, and then continuity.

We revisit and precise the argument in [Ayo14a, Corollaire 11.4] to establish (ii), (iii) and (iv).

Let us first treat the case where S is the spectrum of a field. In that case, for $n \neq 1$, both the source and target of $\nu^{n,1}$ are 0, so the only interesting case is $n = 1$. We have to show that the map

$$\nu^{1,1}_k : k^\times \otimes \mathbb{Q} \to H^{1,1}_{\mathcal{M}}(k)$$

is an isomorphism. By the definition of $\nu^{1,1}$, we have to show that the map

$$k^\times \otimes \mathbb{Q} \simeq \mathbf{DA}^{\text{eff}}(k)(\mathbb{Q}, \mathbb{G}_m \otimes \mathbb{Q}) \to \mathbf{DA}(k)(\mathbb{Q}, \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q}))$$

induced by Σ^∞ is an isomorphism.

Let k^{perf} be a perfect closure of k and $h : \mathbf{Spec}(k^{\text{perf}}) \to \mathbf{Spec}(k)$ be the canonical morphism. In the diagram

$$\begin{tikzcd}
\mathbf{DA}^{\text{eff}}(k)(\mathbb{Q}, \mathbb{G}_m \otimes \mathbb{Q}) \arrow[r] \arrow[d, "h^*"'] & \mathbf{DA}(k)(\mathbb{Q}, \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q})) \arrow[d, "h^*"'] & \\
\mathbf{DA}^{\text{eff}}(k^{\text{perf}})(\mathbb{Q}, \mathbb{G}_m \otimes \mathbb{Q}) \arrow[r] & \mathbf{DA}(k^{\text{perf}})(\mathbb{Q}, h^*\Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q})) \arrow[r, "\sim", "(R_h)_*"'] & \mathbf{DA}(k)(\mathbb{Q}, \Sigma^\infty(\mathbb{G}_m \otimes \mathbb{Q}))
\end{tikzcd}$$

the left square commutes by Lemma 1.2.5 3. The left vertical arrow is an isomorphism because $k^\times \otimes \mathbb{Q} \simeq (k^{\text{perf}})^\times \otimes \mathbb{Q}$ (any element of k^{perf} has a power in k), and the right vertical arrow is an isomorphism by separation for $\mathbf{DA}$.

We are now reduced to the case k perfect. Then we can follow a familiar pattern :comparison with $\mathbf{DM}(k)$ using Theorem 2.2.8, Proposition 2.2.10, then with $\mathbf{DM}^{\text{eff}}(k)$ Voevodsky's cancellation theorem (this is where we need k perfect), and finally the classical computation of weight one effective motivic cohomology [MVW06, Lecture 4].

We now do the general case. We can assume S connected, hence integral.

Let $j : U \to S$ be a non-empty open set, Z its closed complement. We stratify $Z = Z_0 \subset Z_1 \subset \ldots \subset Z_k = \emptyset$ in such a way that for all i, the scheme $(Z_i \setminus Z_{i+1})_{\mathrm{red}}$ is regular and in such a way that $(Z \setminus Z_1)$ contains all points of codimension 1 of Z in S. Then by applying inductively localisation, absolute purity (for the regular pair $(S, (Z_i \setminus Z_{i+1})_{\mathrm{red}})$) and the vanishing result Proposition 3.B.3 (i) and (ii) we see that

- the map $\nu^{0,0} : \mathbb{Q}^{\pi_0(Z\setminus Z_1)} \to H^{0,0}_{\mathcal{M}}(Z \setminus Z_1)$ is an isomorphism,
- the pullback map $H^{n,1}_{\mathcal{M}}(S) \to H^{n,1}_{\mathcal{M}}(U)$ is an isomorphism for $n \neq 1, 2$, and
- there is a short exact sequence

$$0 \to H^{1,1}_{\mathcal{M}}(S) \to H^{1,1}_{\mathcal{M}}(U) \to H^{0,0}_{\mathcal{M}}(Z \setminus Z_1) \to H^{2,1}_{\mathcal{M}}(S) \to H^{2,1}_{\mathcal{M}}(U) \to 0$$

Putting this together with the localisation sequence for $\mathcal{O}^\times$ and Pic, we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_S^\times \otimes \mathbb{Q} \arrow[r] \arrow[d, "\nu_S^{1,1}"'] & \mathcal{O}_U^\times \otimes \mathbb{Q} \arrow[r, "\mathrm{val}"] \arrow[d, "\nu_U^{1,1}"'] \arrow[dr, phantom, "\mathbf{(A)}"] & \mathbb{Q}^{\pi_0(Z\setminus Z_1)} \simeq \bigoplus_{z\in Z^{(1)}} \mathbb{Q}[z] \arrow[r] \arrow[d, "\nu^{0,0}"', "\sim"] \arrow[dr, phantom, "\mathbf{(B)}"] & \mathrm{Pic}(S) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\nu_S^{2,1}"'] & \mathrm{Pic}(U) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\nu_U^{2,1}"'] & 0 \\
0 \arrow[r] & H^{1,1}_{\mathcal{M}}(S) \arrow[r] & H^{1,1}_{\mathcal{M}}(U) \arrow[r] & H^{0,0}_{\mathcal{M}}(Z \setminus Z_1) \arrow[r] & H^{2,1}_{\mathcal{M}}(S) \arrow[r] & H^{2,1}_{\mathcal{M}}(U) \arrow[r] & 0.
\end{tikzcd}$$

We claim that the diagram above is commutative. For the two outer diagrams, this follows from the commutation of u_S with pullbacks in Proposition 3.2.9.

For the commutation of diagrams (A) and (B), we have to do more work, since one arrow is defined explicitely using valuations and line bundle attached to a divisor while the other is defined via the absolute purity isomorphism. Instead of giving a long explicit formula, we prefer to see it as a special case of Déglise's machinery of "residual Riemann-Roch formulas" in [Dé, 4.2.1, 5.5.1]; namely, take the diagram (4.2.1 b) in loc. cit. with $\mathbb{E}$ being algebraic K-theory tensor $\mathbb{Q}$, $\mathbb{F}$ being motivic cohomology with rational coefficients, the morphism ϕ being the Chern character, and then use that $\mathcal{O}^\times(S)_{\mathbb{Q}} \oplus \mathrm{Pic}(S)_{\mathbb{Q}} \subset K_1(S) \otimes \mathbb{Q}$ for S regular, and that the Chern character maps coincide with the maps $\nu^{n,1}$ modulo this identification.

Passing to the limit in the previous commutative diagram over all non-empty open sets, using continuity both for motivic cohomology and for the étale cohomology of $\mathbb{G}_{\mathrm{m}}$, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_S^\times \otimes \mathbb{Q} \arrow[r] \arrow[d, "\nu_S^{1,1}"'] & \kappa(S)^\times \otimes \mathbb{Q} \arrow[r, "\mathrm{val}"] \arrow[d, "\nu_U^{1,1}"'] & \bigoplus_{z\in S^{(1)}} \mathbb{Q}[z] \arrow[r] \arrow[d, equal] & \mathrm{Pic}(S) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\nu_S^{2,1}"'] & \mathrm{Pic}(\kappa(S)) \arrow[r] \arrow[d, "\nu_U^{2,1}"'] & 0 \\
0 \arrow[r] & H^{1,1}_{\mathcal{M}}(S) \arrow[r] & H^{1,1}_{\mathcal{M}}(\kappa(S)) \arrow[r] & \bigoplus_{z\in S^{(1)}} \mathbb{Q}[z] \arrow[r] & H^{2,1}_{\mathcal{M}}(S) \arrow[r] & H^{2,1}_{\mathcal{M}}(\kappa(S)) \arrow[r] & 0.
\end{tikzcd}$$

Using the case of a base field treated above, we see that

- the group $H^{n,1}_{\mathcal{M}}(S)$ vanishes for $n \neq 1, 2$, and
- there is a short exact sequence

$$0 \to H^{1,1}_{\mathcal{M}}(S) \to \kappa(S)^\times \otimes \mathbb{Q} \xrightarrow{\mathrm{val}} \bigoplus_{z\in S^{(1)}} \mathbb{Q}[z] \to H^{2,1}_{\mathcal{M}}(S) \to 0.$$

Using the normality (resp. regularity) of S, this implies $H^{1,1}_{\mathcal{M}}(S) \simeq \mathcal{O}(S)^\times_{\mathbb{Q}}$ and $H^{2,1}_{\mathcal{M}}(S) \simeq \mathrm{Pic}(S)_{\mathbb{Q}}$ and finishes the proof.

□

We finish by giving an example which shows that even for weight zero motivic cohomology on normal (but not regular) schemes, the result can differ from étale cohomology.

Proposition 3.B.5. *Let S be a normal excellent surface. Let $\pi : \tilde{S} \to S$ be a resolution of singularities of S, with $D = \pi^{-1}(p)$ simple normal crossing divisor in $\tilde{S}$. Let $\Gamma = (V, E)$ be the resolution graph of D. Then*

$$H^{n,0}_{\mathcal{M}}(S) \simeq \begin{cases} \mathbb{Q},\ n = 0 \\ H^1(\Gamma, \mathbb{Q}),\ n = 2 \\ 0,\ n \neq 0, 2 \end{cases}$$

while on the other hand

$$D(\mathsf{Sm}/S)(\mathbb{Q}_S, \mathbb{Q}_S[n]) \simeq \begin{cases} \mathbb{Q},\ n = 0 \\ 0,\ n \neq 0 \end{cases}.$$

Proof. The last statement comes from the fact that the étale cohomology of a normal scheme with $\mathbb{Q}$-coefficients is trivial. So we concentrate on the first. For $n \leq 0$, the result follows from 3.B.3, so we assume $n > 0$.

We have the cartesian diagram of schemes:

$$\begin{tikzcd} U \arrow[r, "\tilde{j}"] \arrow[d, equal] & \tilde{S} \arrow[d, "\pi"] & D \arrow[l, "\tilde{i}"'] \arrow[d, "\pi_p"] \\ U \arrow[r, "j"] & S & p \arrow[l, "i"'] \end{tikzcd}$$

Localization yields the long exact sequence:

$$\begin{tikzcd} \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S[n-1]) \arrow[r] & \mathbf{DA}(U)(\mathbb{Q}_U, \mathbb{Q}_U[n-1]) \arrow[r] & \mathbf{DA}(p)(\mathbb{Q}_p, i^!\mathbb{Q}_S[n]) \arrow[r] & \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S[n]) \arrow[d] \\ & & & \mathbf{DA}(U)(\mathbb{Q}_U, \mathbb{Q}_U[n]) \end{tikzcd}$$

By Proposition 3.B.3, this yields an isomorphism $\mathbf{DA}(p)(\mathbb{Q}_p, i^!\mathbb{Q}_S[n]) \simeq \mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S[n])$.

Write $\{D_v\}_{v \in V}$ for the set of irreducible components of D and p_e for the intersection points $D_v \cap D_{v'}$ for $vv' \in E$. We put $Z = \bigcup_{e \in E}\{p_e\}$ and $\mathring{D} = D \setminus Z$. Write $k : \mathring{D} \to E$, $l : Z \to D$. Localization gives a distinguished triangle

$$l_*(\tilde{i} \circ l)^!\mathbb{Q}_{\tilde{S}} \to \tilde{i}^!\mathbb{Q}_{\tilde{S}} \to k_*(\tilde{i} \circ k)^!\mathbb{Q}_{\tilde{S}} \xrightarrow{+}.$$

By the relative purity theorem for **DA** (see [Ayo07a, 1.6.1] and [Ayo14a, Corollaire 3.10]) applied to the regular immersions $\tilde{i} \circ l$ and $\tilde{i} \circ k$, this triangle takes the form:

$$l_*\mathbb{Q}_Z(-2)[-4] \to \tilde{i}^!\mathbb{Q}_{\tilde{S}} \to k_*\mathbb{Q}_{\mathring{D}}(-1)[-2] \xrightarrow{+}$$

So we get the exact sequence:

$$\mathbf{DA}(\mathring{D})(\mathbb{Q}_{\mathring{D}}, \mathbb{Q}_{\mathring{D}}(-2)[n-4]) \to \mathbf{DA}(D)(\mathbb{Q}_D, \tilde{i}^!\mathbb{Q}_{\tilde{S}}[n]) \to \mathbf{DA}(Z)(\mathbb{Q}_Z, \mathbb{Q}_Z(-1)[n-2])$$

By Proposition 3.B.2, the groups on the left and on the right are zero for all $n \in \mathbb{Z}$, so we conclude that $\mathbf{DA}(D)(\mathbb{Q}_D, \tilde{i}^!\mathbb{Q}_{\tilde{S}}[n]) = 0$ for all $n \in \mathbb{Z}$.

Now, the fact that π_U is an isomorphism, colocalization and base change for immersions (see [Ayo07a, 1.4.6]) implies that $\mathrm{Cone}(i^!\mathbb{Q}_S \to \pi_{p,*}\tilde{i}^!\mathbb{Q}_{\tilde{S}}) \simeq \mathrm{Cone}(\mathbb{Q}_p \to \pi_{p,*}\mathbb{Q}_D)$. Combining with the previous result, we get that for all $n \in \mathbb{Z}$:

$$\mathbf{DA}(S)(\mathbb{Q}_S, \mathbb{Q}_S[n]) \simeq \mathbf{DA}(p)(\mathbb{Q}_p, \mathrm{Cone}(\mathbb{Q}_p \to \pi_{p,*}\mathbb{Q}_D)[n-1])) \simeq \mathbf{DA}(p)(\mathbb{Q}_p, \pi_{p,*}\mathbb{Q}_D[n-1])$$

(The last isomorphism comes because $n > 1$).

Using Cech descent for closed covers and Proposition 3.B.2, it is then easy to see that this last group is isomorphic to $\mathbb{Q}$ if $n = 0$ (note that Γ is connected by normality of S), isomorphic to $H^1(\Gamma, \mathbb{Q})$ if $n = 1$, and 0 otherwise.

$\square$

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